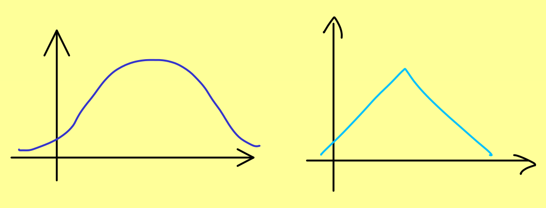
Distributions – Part 13

convolution $*$ for integrable functions $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $\underbrace{\int_{\mathbb{R}^n} |f(x)| \, dx}_{\|f\|_1} < \infty$

$f \in \mathcal{L}^1(\mathbb{R}^n)$

($f \in L^1(\mathbb{R}^n)$ for equivalence classes)



for f, g define:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y) \, dy$$ exists almost everywhere for $x \in \mathbb{R}^n$

One has: $\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$ ⟿ $L^1(\mathbb{R}^n)$ with $+$ and $*$ is an algebra over $\mathbb{R}$

Generalizations: $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$

$$\langle \psi * f, \varphi \rangle = \int_{\mathbb{R}^n} (\psi * f)(x)\, \varphi(x) \, dx$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \psi(x-y)\, f(y) \, dy \right) \varphi(x) \, dx$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} \underbrace{\psi(x-y)}_{\check{\psi}(y-x)} \varphi(x) \, dx \right) dy \quad \text{with } \check{\psi}(z) = \psi(-z)$$

$$= \int_{\mathbb{R}^n} f(y) \, (\check{\psi} * \varphi)(y) \, dy = \langle f, \check{\psi} * \varphi \rangle$$

For regular distributions: $\langle T_{\psi * f}, \varphi \rangle = \langle T_f, \check{\psi} * \varphi \rangle$

Definition: For $T \in \mathcal{D}'(\mathbb{R}^n)$, $\psi \in \mathcal{D}(\mathbb{R}^n)$ define a distribution:

$$\langle \psi * T, \varphi \rangle := \langle T, \check{\psi} * \varphi \rangle$$

convolution: $* : \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ bilinear map