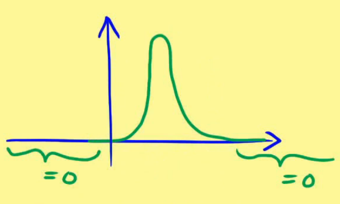
The Bright Side of Mathematics

Distributions - Part 2

Testfunctions

$\varphi: \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$)



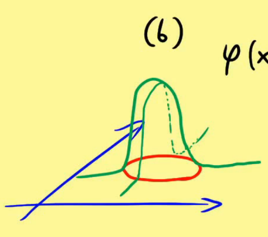
Space of test functions: $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$

- $\mathcal{D}(\mathbb{R}^n)$: vector space + specific convergence (topology / metric)
- $C_c^\infty(\mathbb{R}^n)$: differentiable, arbitrarily often; compact support

Examples: $\varphi: \mathbb{R}^n \to \mathbb{R}$

(a) $\varphi = 0$

(b) $\varphi(x) = \begin{cases} 0 & , \|x\| \geq 1 \\ \exp\left(\frac{-1}{1-\|x\|^2}\right) & , \|x\| < 1 \end{cases}$



$\varphi: \mathbb{R}^2 \to \mathbb{R}$

Notations:

- support of a function $\varphi: \mathbb{R}^n \to \mathbb{R}$:
  $\mathrm{supp}(\varphi) = \overline{\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}}$ ← closure in $\mathbb{R}^n$!
- for $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N}_0$, we call $\alpha = (\alpha_1, \alpha_n, \dots, \alpha_n)$ a multi-index

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

Example: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = 2x_1^2 x_2^3$, $\alpha = (2,1)$

$$(D^\alpha f)(x_1, x_2) = \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial}{\partial x_2} 2x_1^2 x_2^3\right) = \frac{\partial^2}{\partial x_1^2}\left(6x_1^2 x_2^2\right) = 12 x_2^2$$

This means: $\varphi \in C^\infty(\mathbb{R}^n) \iff D^\alpha \varphi \in C(\mathbb{R}^n)$ for all multi-indices α