The Bright Side of Mathematics

The following pages cover the whole Distributions course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Distributions - Part 1

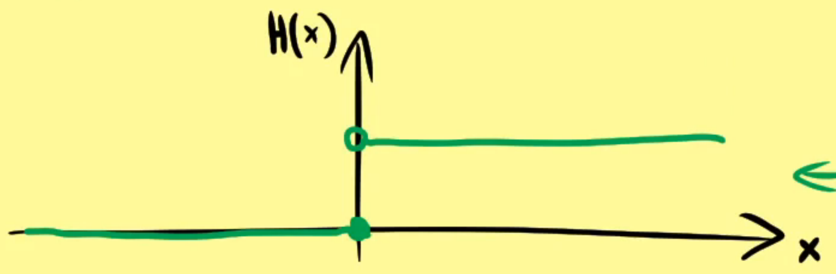
Analysis: Function, limit, derivative

Differential equations, Fourier series, Fourier transform

$\hookrightarrow$ solutions with sharp turns

$\hookrightarrow$ more general solutions

Historical example: 1927 Paul Dirac, $H: \mathbb{R} \to \mathbb{R}$
Heaviside function



$\leftarrow$ Derivative?

- classical derivative has a problem at $x = 0$
- "more general" derivative should be "delta function" δ!

$\delta(x) = 0$ for $x \neq 0$, and

for $\varepsilon > 0$: $\displaystyle\int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx = \int_{-\varepsilon}^{\varepsilon} H'(x)\,dx = H(\varepsilon) - H(-\varepsilon) = 1 - 0 = 1$

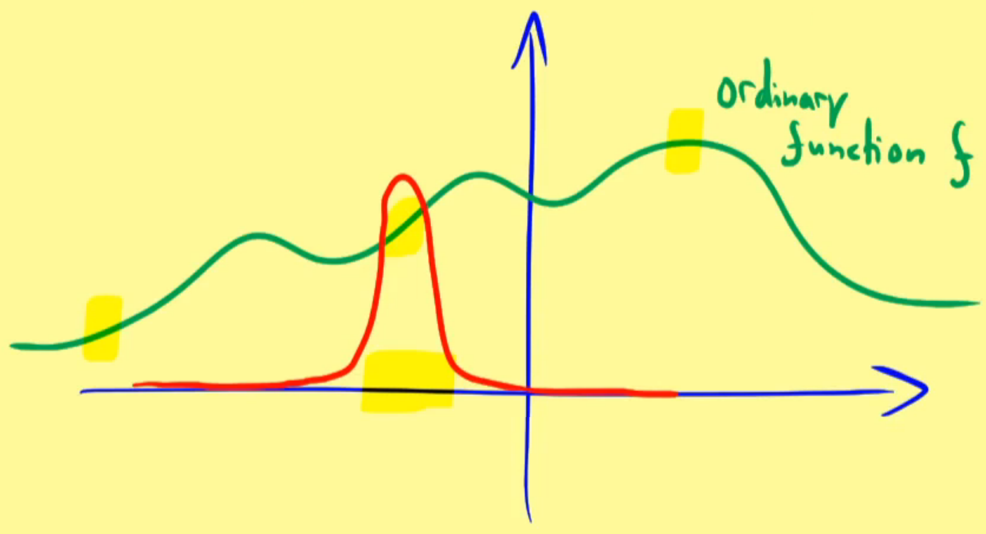
- δ can't be an ordinary function

$\delta(x) = 0$ almost everywhere (w.r.t. Lebesgue measure)

hence $\displaystyle\int_{-\varepsilon}^{\varepsilon} \delta(x) = 0 \neq 1$

- Dirac wants to calculate with $\underbrace{\delta', \delta'', \delta'''}_{\to \text{ meaning?}} \ldots$

$\Rightarrow$ δ will be defined as a distribution $\to$ "generalised function"



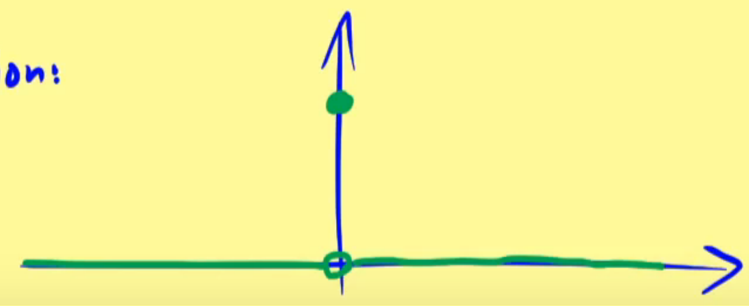
Use so-called test functions
$\varphi: \mathbb{R} \to \mathbb{R}$

$$\varphi \mapsto \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx \in \mathbb{R}$$

linear map

$$\varphi \mapsto \varphi(0)$$

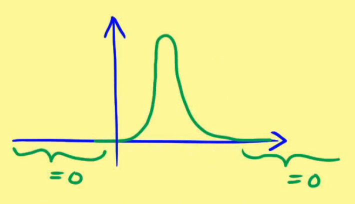
Delta function:



Distributions – Part 2

Testfunctions

$\varphi: \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$)



Space of test functions: $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$

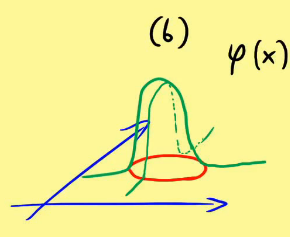
vector space + specific convergence (topology / metric)

- differentiable, arbitrarily often
- compact support

Examples: $\varphi: \mathbb{R}^n \to \mathbb{R}$

(a) $\varphi = 0$

(b) $\varphi(x) = \begin{cases} 0 & , \|x\| \geq 1 \\ \exp\left(\frac{-1}{1-\|x\|^2}\right) & , \|x\| < 1 \end{cases}$



$\varphi: \mathbb{R}^2 \to \mathbb{R}$

Notations: – support of a function $\varphi: \mathbb{R}^n \to \mathbb{R}$:

$$\mathrm{supp}(\varphi) = \overline{\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}}$$ ← closure in $\mathbb{R}^n$!

– for $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N}_0$, we call $\alpha = (\alpha_1, \alpha_n, \ldots, \alpha_n)$ a multi-index

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

Example: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = 2x_1^2 x_2^3$, $\alpha = (2,1)$

$$(D^\alpha f)(x_1, x_2) = \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial}{\partial x_2} 2x_1^2 x_2^3\right) = \frac{\partial^2}{\partial x_1^2}\left(6x_1^2 x_2^2\right) = 12 x_2^2$$

This means: $\varphi \in C^\infty(\mathbb{R}^n) \iff D^\alpha \varphi \in C(\mathbb{R}^n)$ for all multi-indices α

Distributions – Part 3

Convergence for test functions

$$\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$$ (space of test functions)

Notions of convergence: • If one has a norm $\|\cdot\|$:

$$f_n \longrightarrow f \quad :\Longleftrightarrow \quad \|f_n - f\| \xrightarrow{n\to\infty} 0$$

• If one has a metric $d(\cdot,\cdot)$:

$$f_n \longrightarrow f \quad :\Longleftrightarrow \quad d(f_n, f) \xrightarrow{n\to\infty} 0$$

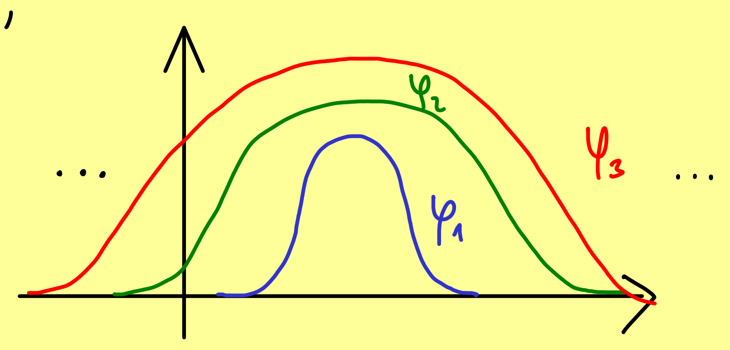
$\Longrightarrow$ Both notions are not sufficient here $\rightsquigarrow$ very strong notion needed

Definition: For $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we write

$$\varphi_k \xrightarrow[\mathcal{D}]{} \varphi \qquad \text{if}$$

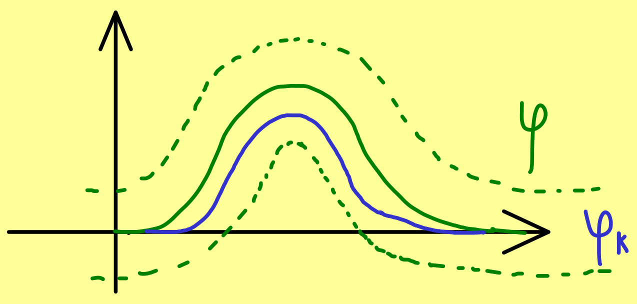
(a) There is a bounded set M, such that outside of it: $\varphi_k = 0$ for all k.

Not:



(b) Uniform convergence $\varphi_k \xrightarrow[\text{unif.}]{} \varphi$ and for all multi-indices α

$$D^\alpha \varphi_k \xrightarrow[\text{unif.}]{} D^\alpha \varphi$$



Use the supremum norm $\|f\|_\infty := \sup\{\, |f(x)| \mid x \in \mathbb{R}^n \}$.

$$\varphi_k \xrightarrow[\mathcal{D}]{} \varphi \quad \Longleftrightarrow \quad$$

(a) $\exists\, C \subseteq \mathbb{R}^n$ compact such that $\operatorname{supp}(\varphi_1), \operatorname{supp}(\varphi_2), \ldots \subseteq C$

(b) $\forall \alpha$ multi-index we have $\|D^\alpha \varphi_k - D^\alpha \varphi\|_\infty \xrightarrow{k\to\infty} 0$

Distributions - Part 4

Space of distributions

$\mathcal{D}(\mathbb{R}^n)$ - vector space of test functions with notion of convergence for sequences of test functions (special case of a topological vector space)

$T: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is called distribution if

- T is linear $\left(\begin{array}{l} T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2) \\ T(\lambda \cdot \varphi) = \lambda \cdot T(\varphi) \end{array} \right)$
- T is continuous in the following sense:

  For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and all sequences $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with $\varphi_k \xrightarrow[\mathcal{D}]{} \varphi$:

  $$T(\varphi_k) \xrightarrow{k \to \infty} T(\varphi) \quad \text{(sequentially continuous)}$$

Notation: $\mathcal{D}(\mathbb{R}^n)'$ or $\mathcal{D}'(\mathbb{R}^n)$ space of distributions

important!

elements: $T: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$

Examples: (a) delta distribution: $\delta: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$

$$\delta(\varphi) = \varphi(0)$$

- linear ✓
- continuous: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with $\varphi_k \xrightarrow[\mathcal{D}]{} \varphi$ (in particular: $\varphi_k(x) \xrightarrow{k \to \infty} \varphi(x)$ for all $x \in \mathbb{R}^n$),

  We have: $\delta(\varphi_k) = \varphi_k(0) \xrightarrow{k \to \infty} \varphi(0) = \delta(\varphi)$ ✓

(b) Continuous functions "are" distributions:

For $f \in C(\mathbb{R}^n)$ define: $T_f : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$$

- linear ✓
- continuous: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with $\varphi_k \xrightarrow[\mathcal{D}]{} \varphi$ (in particular: $\varphi_k \xrightarrow[\text{uniform convergence}]{} \varphi$),

  We have $T_f(\varphi_k) \xrightarrow{k \to \infty} T_f(\varphi)$ ✓

Important property: $f, g \in C(\mathbb{R}^n)$, $f \neq g \implies T_f \neq T_g$

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Distributions – part 5

Regular distributions

Proposition: $T: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) linear. Then:

T is a distribution $\Longleftrightarrow \forall_{K \subseteq \mathbb{R}^n \text{ compact}} \; \exists_{m \in \mathbb{N}_0} \; \exists_{C > 0} \; \forall_{\varphi \in \mathcal{D}(\mathbb{R}^n)}$ $\operatorname{supp}(\varphi) \subseteq K \Longrightarrow |T(\varphi)| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$

Proof: $(\Leftarrow)$ Let $\varphi_k, \varphi \in \mathcal{D}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ with $\varphi_k \xrightarrow{\mathcal{D}} \varphi$.

Then there is a $K \subseteq \mathbb{R}^n$ with $\operatorname{supp}(\varphi_k) \subseteq K$

and for all α we have $\|D^\alpha \varphi_k - D^\alpha \varphi\|_\infty \xrightarrow{k \to \infty} 0$.

$$|T(\varphi_k) - T(\varphi)| = |T(\varphi_k - \varphi)| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi_k - D^\alpha \varphi\|_\infty \xrightarrow{k \to \infty} 0$$

$(\Rightarrow)$ Proof by contraposition:

$\exists_{K \subseteq \mathbb{R}^n \text{ compact}} \; \forall_{m \in \mathbb{N}_0} \; \forall_{C > 0} \; \exists_{\varphi \in \mathcal{D}(\mathbb{R}^n)}$ $\operatorname{supp}(\varphi) \subseteq K$ and $|T(\varphi)| > C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$

For $C = m = k \in \mathbb{N}$ take $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$ with $|T(\varphi_k)| > k \cdot \sum_{\underbrace{|\alpha|}_{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \leq k} \|D^\alpha \varphi_k\|_\infty$

Define: $\psi_k(x) := \dfrac{1}{|T(\varphi_k)|} \varphi_k(x)$. Then: $\psi_k \xrightarrow{\mathcal{D}} 0$

But: $|T(\psi_k)| = \dfrac{1}{|T(\varphi_k)|} \cdot |T(\varphi_k)| = 1 \;\not\xrightarrow{k \to \infty}\; 0$ $\square$

Definition: $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is called locally integrable

if for all compact $K \subseteq \mathbb{R}^n$: $\int_K |f(x)| \, dx < \infty$

Then we write: $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$.

For example: $f: \mathbb{R} \to \mathbb{R}, \; x \mapsto x^2 \Rightarrow f \in \mathcal{L}^1_{loc}(\mathbb{R})$

For $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, define $T_f \in \mathcal{D}'(\mathbb{R}^n)$ by $T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\, \varphi(x) \, dx$

$$\boxed{|T(\varphi)| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty}$$

$$|T_f(\varphi)| \leq \int_{\operatorname{supp}(\varphi)} |f(x)| \cdot \underbrace{|\varphi(x)|}_{\leq \|\varphi\|_\infty} dx \leq \int_K |f(x)| \, dx \cdot \|\varphi\|_\infty \quad ✓$$

K ← compact set $K \supseteq \operatorname{supp}(\varphi)$

Definition: $T \in \mathcal{D}'(\mathbb{R}^n)$ is called regular if there is a locally integrable function f such that $T = T_f$.

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Distributions – part 6

Delta distribution is not regular:

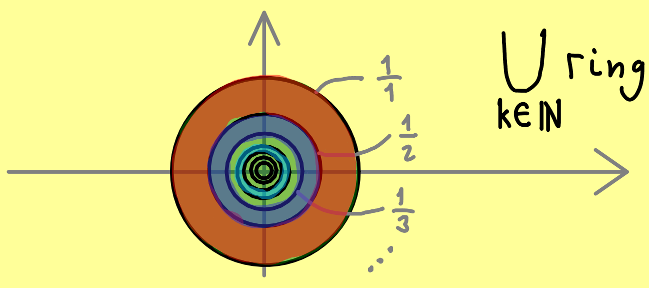
There is no locally integrable function $f: \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$)

with $\delta(\varphi) = T_f(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\underbrace{\delta(\varphi)}_{\varphi(0)} = \underbrace{T_f(\varphi)}_{\int_{\mathbb{R}^n} f(x)\varphi(x)\,dx}$$

Proof: Assume there is $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ with $\varphi(0) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(1) $\int_{\|x\| \leq 1} |f(x)|\,dx = a < \infty$



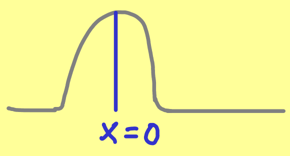
$\bigcup_{k \in \mathbb{N}} \text{ring}_k = \text{disc} \setminus \{0\}$ (disjoint union!)

$$\int_{\|x\| \leq 1} |f(x)|\,dx = \int_{\bigcup_{k\in\mathbb{N}} \text{ring}_k} |f(x)|\,dx \overset{\text{(measure theory/ integration theory)}}{=} \sum_{k=1}^{\infty} \int_{\text{ring}_k} |f(x)|\,dx \implies \exists k_0 \in \mathbb{N} : \sum_{k=k_0}^{\infty} \int_{\text{ring}_k} |f(x)|\,dx \leq \frac{1}{2}$$

In summary: There is $\varepsilon > 0$ with $\int_{\|x\|\leq\varepsilon} |f(x)|\,dx = b \leq \frac{1}{2}$

(2) Take test function:

$$\varphi_\varepsilon(x) = \begin{cases} 0 & , \|x\| \geq \varepsilon \\ \exp\left(-\frac{1}{1-\left(\frac{\|x\|}{\varepsilon}\right)^2}\right) & , \|x\| < \varepsilon \end{cases}$$



$$\varphi_\varepsilon(0) = \left| \int_{\mathbb{R}^n} f(x)\varphi_\varepsilon(x)\,dx \right| \leq \int_{\|x\|\leq\varepsilon} |f(x)| \cdot |\varphi_\varepsilon(x)|\,dx \leq \underbrace{\|\varphi_\varepsilon\|_\infty}_{\varphi_\varepsilon(0)} \cdot \underbrace{\int_{\|x\|\leq\varepsilon} |f(x)|\,dx}_{b}$$

$$\leq \varphi_\varepsilon(0) \cdot \frac{1}{2} \implies \text{contradiction}$$

Distributions – Part 7

$f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ ⟵ vector space of functions

$T_f \in \mathcal{D}'(\mathbb{R}^n)$ ⟵ vector space of distributions?

Fact: $\mathcal{D}'(\mathbb{R}^n)$ is a real (or complex) vector space:

- addition: $+$ for $T, S \in \mathcal{D}'(\mathbb{R}^n)$, define $T + S \in \mathcal{D}'(\mathbb{R}^n)$

$$(T+S)(\varphi) = T(\varphi) + S(\varphi)$$

$$(T_f + T_g)(\varphi) = T_f(\varphi) + T_g(\varphi)$$

$$= \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx + \int_{\mathbb{R}^n} g(x)\,\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n} \big(f(x) + g(x)\big)\varphi(x)\,dx = T_{f+g}(\varphi)$$

- scalar multiplication: $\cdot$ for $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $T \in \mathcal{D}'(\mathbb{R}^n)$

define $\lambda \cdot T$ by:

$$(\lambda \cdot T)(\varphi) = \lambda \cdot T(\varphi)$$

(we have all calculations rules in a vector space)

duality pairing: $\langle T, \varphi \rangle := T(\varphi)$

$\langle \cdot , \cdot \rangle : \mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) bilinear map

Distributions – Part 8

$T, S \in \mathcal{D}'(\mathbb{R}^n) \rightsquigarrow T \cdot S$ makes problems...

Multiplication with smooth functions: $S \in \mathcal{D}'(\mathbb{R}^n)$, $f \in C^\infty(\mathbb{R}^n)$

$T_f \cdot S$ can be defined as a new distribution.

First case: S is a regular distribution, $S = T_g$ with $g \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$

$$(T_f \cdot T_g)(\varphi) \overset{\text{should be}}{=} T_{f \cdot g}(\varphi) = \int_{\mathbb{R}^n} (f(x) \cdot g(x)) \varphi(x) \, dx$$

$$= \int_{\mathbb{R}^n} g(x) (f(x) \varphi(x)) \, dx = T_g(f \cdot \varphi) \quad \text{with } f \cdot \varphi \in \mathcal{D}(\mathbb{R}^n)$$

Definition: $T_f \cdot S$ or $f \cdot S$ for $f \in C^\infty(\mathbb{R}^n)$ is the distribution defined by:

$$\langle f \cdot S, \varphi \rangle := \langle S, f \cdot \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n)$$

Proof: (1) $f \cdot S : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is linear ✓

(2) Leibniz rule: $D^\alpha (f \cdot \varphi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^\beta f) \cdot (D^{\alpha - \beta} \varphi)$

$$\boxed{S \text{ is a distribution} \iff \underset{K \subseteq \mathbb{R}^n \text{ compact}}{\forall} \ \underset{m \in \mathbb{N}_0}{\exists} \ \underset{C > 0}{\exists} \ \underset{\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^n)}{\forall} \ \operatorname{supp}(\tilde{\varphi}) \subseteq K \Rightarrow |S(\tilde{\varphi})| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \tilde{\varphi}\|_\infty}$$

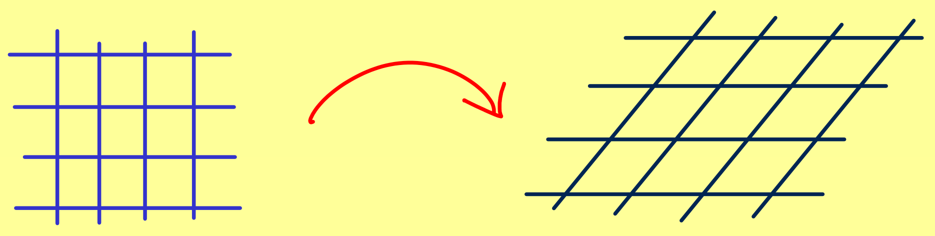
$$|(f \cdot S)(\varphi)| = |S(f \varphi)| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha (f \varphi)\|_\infty$$

$$\leq C \cdot \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} \|(D^\beta f)\|_\infty \, \|D^{\alpha - \beta} \varphi\|_\infty \binom{\alpha}{\beta}$$

$$\leq \tilde{C} \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$$

Distributions – Part 9

invertible linear map $A : \mathbb{R}^n \to \mathbb{R}^n$



$$f \in \mathcal{L}^1_{loc}(\mathbb{R}^n) \quad \Rightarrow \quad f \circ A \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$$

$$\langle T_{f\circ A}, \varphi \rangle = \int_{\mathbb{R}^n} f(Ax)\varphi(x)\, dx = \frac{1}{|\det(A)|} \int_{\mathbb{R}^n} f(\underbrace{Ax}_{y})\varphi(x) \underbrace{|\det(A)|\, dx}_{dy}$$

$$= \frac{1}{|\det(A)|} \int_{\mathbb{R}^n} f(y)\varphi(A^{-1}y)\, dy = \left\langle T_f, \frac{1}{|\det(A)|} \varphi \circ A^{-1} \right\rangle$$

Definition: Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map.

Define: $\langle T \circ A, \varphi \rangle := \left\langle T, \frac{1}{|\det(A)|} \varphi \circ A^{-1} \right\rangle$

Strange notation: $\delta(x)$ denotes the delta distribution

Or: (with strange notation) $\langle T(Ax), \varphi(x) \rangle := \left\langle T(x), \frac{1}{|\det(A)|} \varphi(A^{-1}x) \right\rangle$

For $b \in \mathbb{R}^n$ define: $\langle T(Ax+b), \varphi(x) \rangle := \left\langle T(x), \frac{1}{|\det(A)|} \varphi(A^{-1}(x-b)) \right\rangle$

and: $G^{-1} \in C^\infty(\mathbb{R}^n)$

For $G \in C^\infty(\mathbb{R}^n)$ bijective, define: $\langle T(Gx), \varphi(x) \rangle := \left\langle T(x), \frac{1}{|\det(\underbrace{J_G(x)}_{\text{Jacobian matrix of } G})|} \varphi(G^{-1}x) \right\rangle$

Example: $A : \mathbb{R}^n \to \mathbb{R}^n$ rotation $\left(A^{-1} = A^T, \ |\det(A)| = 1 \right)$

$$\langle \delta(Ax), \varphi(x) \rangle = \langle \delta(x), \varphi(A^{-1}x) \rangle = \varphi(A^{-1}0) = \varphi(0)$$

$$= \langle \delta(x), \varphi(x) \rangle \Rightarrow \delta(Ax) = \delta(x)$$

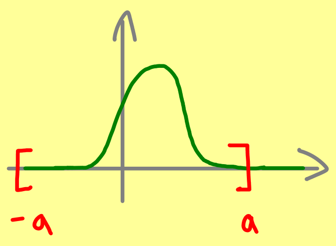
$\Rightarrow$ delta distribution is rotational invariant

Distributions – Part 10

Motivation: $f \in C^1(\mathbb{R}^n)$ $(n=1)$

We get two regular distributions: $T_f\,,\; T_{f'} \in \mathcal{D}'(\mathbb{R}^n)$

We have: $\langle T_{f'}, \varphi \rangle = \int_{\mathbb{R}} f'(x)\,\varphi(x)\,dx$ $\quad \mathbb{R} \leftarrow [-a,a] \supseteq \mathrm{supp}(\varphi)$



$$= \int_{-a}^{a} f'(x)\,\varphi(x)\,dx$$

$$= \underbrace{f(x)\,\varphi(x)\Big|_{-a}^{a}}_{=0} - \int_{-a}^{a} f(x)\,\varphi'(x)\,dx$$

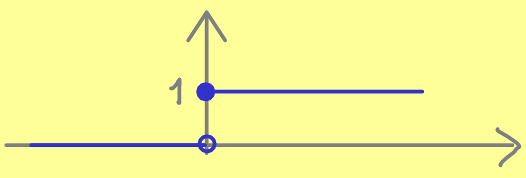
$$= \langle -T_f, \varphi' \rangle$$

Definition: For a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$, we define a new distribution $D^\alpha T \in \mathcal{D}'(\mathbb{R}^n)$ (for any multi-index α), called the (distributional) partial derivative of T, by:

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$$

Note: $D^\alpha(T_f) = T_{D^\alpha f}$ for $f \in C^\infty(\mathbb{R}^n)$

Example: (a) Heaviside function $H : \mathbb{R} \longrightarrow \mathbb{R}$
$\alpha = (1)$ $\qquad (n=1)$



$$\langle D^\alpha (T_H), \varphi \rangle = (-1)^1 \langle T_H, \varphi' \rangle$$

$$= -\int_{\mathbb{R}} H(x)\,\varphi'(x)\,dx = -\int_{-a}^{a} H(x)\,\varphi'(x)\,dx$$

$\mathbb{R} \leftarrow [-a,a] \supseteq \mathrm{supp}(\varphi') \supseteq \mathrm{supp}(\varphi)$

$$= -\int_0^a 1 \cdot \varphi'(x)\,dx = -\int_0^a \varphi'(x)\,dx$$

$$= \underbrace{-\varphi(a)}_{=0} + \varphi(0) = \langle \delta, \varphi \rangle$$

distributional derivative of H $= \delta$

(b) $n=1$, $\alpha=(1)$ $\quad \langle D^\alpha \delta, \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$

distributional derivative of δ

Distributions – Part 11

$$T \in \mathcal{D}'(\mathbb{R}^n) \implies D^\alpha T \in \mathcal{D}'(\mathbb{R}^n) \quad \text{(for any multi-index } \alpha\text{)}$$

Therefore: $D^\alpha : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

- linear
- continuous

Result: For distributions T_k $(k \in \mathbb{N})$, we have:

$$D^\alpha \left(\sum_{k=1}^{\infty} T_k \right) = \sum_{k=1}^{\infty} D^\alpha T_k$$

Example: Laplace's equation: $\Delta T = 0$ $\quad \left(\Delta = D^{\overset{\alpha}{=}(2,0,0)} + D^{\overset{\beta}{=}(0,2,0)} + D^{\overset{\gamma}{=}(0,0,2)} \right)$

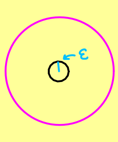
$n = 3$

$$\hookrightarrow \; \gamma(x) = -\frac{1}{4\pi} \cdot \frac{1}{\|x\|}$$

euclidean/standard norm in $\mathbb{R}^3$

regular distribution: T_γ, $\quad \gamma \in \mathcal{L}^1_{loc}(\mathbb{R}^3)$

$$\langle \Delta T_\gamma , \varphi \rangle = \langle T_\gamma , \Delta \varphi \rangle = \int_{B_R(0)} \gamma(x)\, \Delta\varphi(x)\, dx$$



$$= \int_{B_R(0) \setminus B_\varepsilon(0)} \gamma(x)\, \Delta\varphi(x)\, dx + \int_{B_\varepsilon(0)} \gamma(x)\, \Delta\varphi(x)\, dx$$

$$= \lim_{\varepsilon \to 0^+} \int_{B_R(0) \setminus B_\varepsilon(0)} \gamma(x)\, \Delta\varphi(x)\, dx$$

Use Green's identities!

$$= \varphi(0) = \langle \delta , \varphi \rangle$$

$$\implies \Delta T_\gamma = \delta \qquad \text{(fundamental solution)}$$

Definition: For a differential operator $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$, $\quad (P(D) = 0)$

we call $T \in \mathcal{D}'(\mathbb{R}^n)$ a fundamental solution if

$$P(D)\, T = \delta$$

Distributions – Part 12

$T \in \mathcal{D}'(\mathbb{R}^n)$, $\underset{\text{(compact)}}{K \subseteq \mathbb{R}^n}$. There is $m \in \mathbb{N}_0$, $c > 0$ such that:

$$\text{supp}(\varphi) \subseteq K \quad \Rightarrow \quad |\langle T, \varphi \rangle| \leq c \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$$

$$\leq \tilde{c} \cdot \underbrace{\max\left\{ |D^\alpha \varphi(x)| \;\middle|\; x \in \mathbb{R}^n , |\alpha| \leq m \right\}}_{\|\varphi\|_m}$$

Definition: $T \in \mathcal{D}'(\mathbb{R}^n)$ is called a distribution of finite order m if:

$$\underset{m \in \mathbb{N}_0}{\exists} \quad \underset{\substack{K \subseteq \mathbb{R}^n \\ \text{compact}}}{\forall} \quad \underset{c > 0}{\exists} \quad \underset{\varphi \in \mathcal{D}(\mathbb{R}^n)}{\forall} \quad \text{supp}(\varphi) \subseteq K \Rightarrow |\langle T, \varphi \rangle| \leq c \cdot \|\varphi\|_m$$

Regular distribution: $|\langle T_f, \varphi \rangle| = \left| \int_K f(x)\, \varphi(x)\, dx \right| \leq \underbrace{\int_K |f(x)|\, dx}_{c} \cdot \|\varphi\|_0$

$\Rightarrow$ of order 0

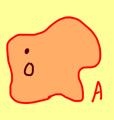
Theorem: $\{ T : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{C} \mid T \text{ is of order } 0 \}$

$$\xleftrightarrow{\text{bijection}}$$

$\{ \mu : \mathcal{B}(\mathbb{R}^n) \longrightarrow \mathbb{C} \cup \{\infty\} \mid \mu \text{ complex Radon measure} \}$

For μ define: $\langle T_\mu , \varphi \rangle := \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x)$

Example: Dirac measure: $\delta(A) := \begin{cases} 0 , & 0 \notin A \\ 1 , & 0 \in A \end{cases}$



Corresponding distribution: $\langle T_\delta , \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x)\, d\delta(x) = \varphi(0)$

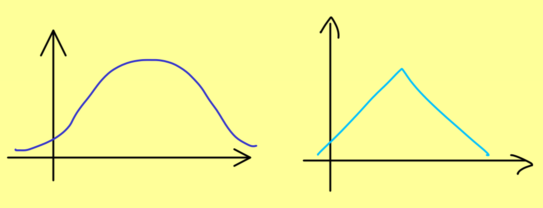
$\Rightarrow$ T_δ is the delta distribution

Distributions – Part 13

convolution $*$ for integrable functions $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $\overbrace{\int_{\mathbb{R}^n} |f(x)|\, dx}^{\|f\|_1} < \infty$

$\underbrace{\qquad\qquad\qquad}_{f \in \mathcal{L}^1(\mathbb{R}^n)}$

($f \in L^1(\mathbb{R}^n)$ for equivalence classes)



for f, g define:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$ exists almost everywhere for $x \in \mathbb{R}^n$

One has: $\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$ $\rightsquigarrow$ $L^1(\mathbb{R}^n)$ with $+$ and $*$ is an algebra over $\mathbb{R}$

Generalizations: $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$

$$\langle \psi * f, \varphi \rangle = \int_{\mathbb{R}^n} (\psi * f)(x)\, \varphi(x)\, dx$$

$$= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \psi(x-y)\, f(y)\, dy \Big)\, \varphi(x)\, dx$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f(y) \Big(\int_{\mathbb{R}^n} \underbrace{\psi(x-y)}_{\check{\psi}(y-x)}\, \varphi(x)\, dx \Big)\, dy$$

with $\check{\psi}(z) = \psi(-z)$

$$= \int_{\mathbb{R}^n} f(y)\, (\check{\psi} * \varphi)(y)\, dy = \langle f, \check{\psi} * \varphi \rangle$$

For regular distributions: $\langle T_{\psi * f}, \varphi \rangle = \langle T_f, \check{\psi} * \varphi \rangle$

Definition: For $T \in \mathcal{D}'(\mathbb{R}^n)$, $\psi \in \mathcal{D}(\mathbb{R}^n)$ define a distribution:

$$\langle \psi * T, \varphi \rangle := \langle T, \check{\psi} * \varphi \rangle$$

convolution: $* : \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ bilinear map

Distributions – Part 14

convolution: $* : \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

$$\langle \psi * \delta , \varphi \rangle = \langle \delta , \check{\psi} * \varphi \rangle \qquad \text{with } \check{\psi}(z) = \psi(-z)$$

$$= (\check{\psi} * \varphi)(0)$$

$$= \int_{\mathbb{R}^n} \check{\psi}(0-y)\, \varphi(y)\, dy$$

$$= \int_{\mathbb{R}^n} \psi(y)\, \varphi(y)\, dy = \langle T_\psi , \varphi \rangle$$

Hence: $\psi * \delta = \psi$ for all $\psi \in \mathcal{D}(\mathbb{R}^n)$

seen as a regular distribution

neutral element for $*$

Properties: (a) For all multi-indices α :

$$D^\alpha(\psi * T) = (D^\alpha \psi) * T = \psi * (D^\alpha T)$$

(b) $\psi_1 * (\psi_2 * T) = (\psi_1 * \psi_2) * T$

Application: differential operator: $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$

fundamental solution: $P(D)E = \delta , \quad E \in \mathcal{D}'(\mathbb{R}^n)$

partial differential equation: $P(D)u = f \rightsquigarrow$ search for u

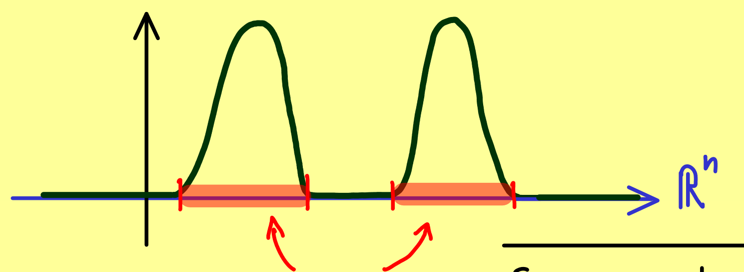
$(\Delta u = f)$

How about $u = f * E$?

$$P(D)u = P(D)(f * E) = f * (\underbrace{P(D)E}_{\delta}) = f$$

Distributions - Part 15

Support: $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

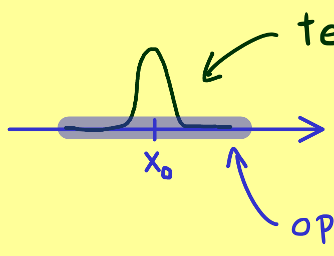


$\mathrm{supp}(f) := \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$

complement is the largest open set $U \subseteq \mathbb{R}^n$ such that $f|_U = 0$

Local behaviour of a distribution? $T \in \mathcal{D}'(\mathbb{R}^n)$ What is the value of T at a point $x_0 \in \mathbb{R}^n$?

not meaningful



$T = 0$ in U $:\Longleftrightarrow$ $T(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\mathrm{supp}(\varphi) \subseteq U$

Example: $\delta \in \mathcal{D}'(\mathbb{R}^n)$: $\delta = 0$ in $\mathbb{R}^n \setminus \{0\}$ (since $\delta(\varphi) = \varphi(0)$)

support of δ is given by $\{0\}$

Proposition: For $T \in \mathcal{D}'(\mathbb{R}^n)$, there a maximal open set $U_{max} \subseteq \mathbb{R}^n$ with $T = 0$ in U_{max}.

The complement is called the support of T :

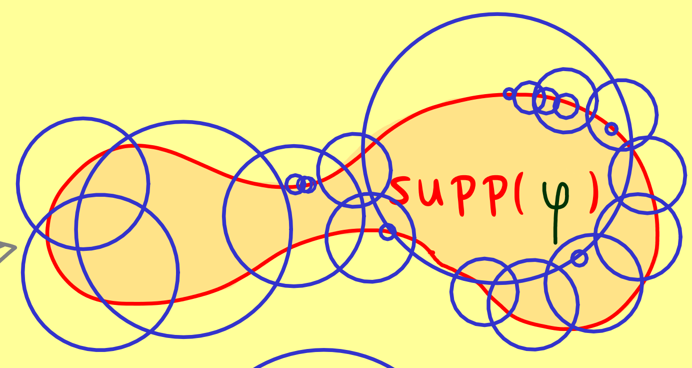
$$\mathrm{supp}(T) := \mathbb{R}^n \setminus U_{max} \quad \text{(closed set)}$$

Proof: Define: $\mathcal{U} := \{U \subseteq \mathbb{R}^n \text{ open} \mid T = 0 \text{ in } U\}$

$U_{max} := \bigcup_{U \in \mathcal{U}} U$ Question: Do we have $T = 0$ in U_{max} ?

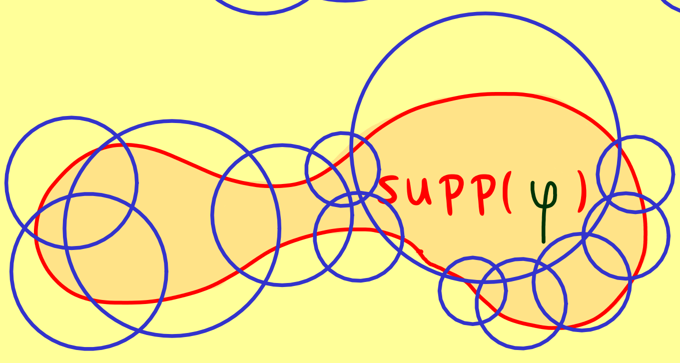
Take $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\mathrm{supp}(\varphi) \subseteq U_{max}$

$\bigcup_{U \in \mathcal{U}} U$ covering



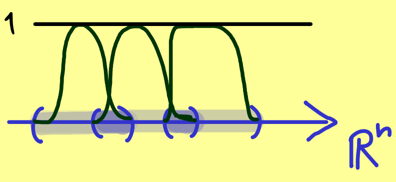
supp(φ) is compact

$\Longrightarrow$ we have a finite subcover



$$\mathrm{supp}(\varphi) \subseteq U_1 \cup U_2 \cup \dots \cup U_m$$

Partition of unity: There are test functions $\psi_1, \psi_2, \dots, \psi_m \in \mathcal{D}(\mathbb{R}^n)$ with $\mathrm{supp}(\psi_i) \subseteq U_i$ such that:



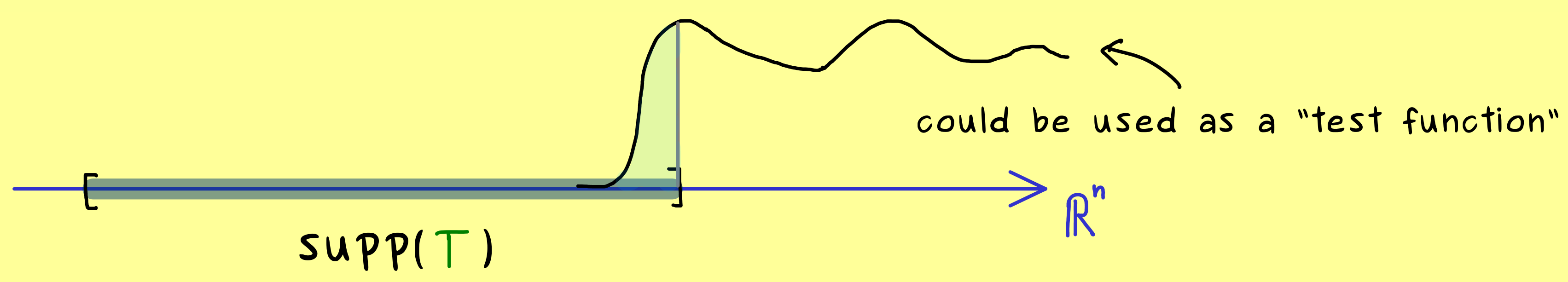
$$1 = \sum_{i=1}^{m} \psi_i(x) \quad \text{for all } x \in \mathrm{supp}(\varphi)$$

partition of unity
$\Longrightarrow$ $\varphi(x) = \sum_{i=1}^{m} \psi_i(x) \cdot \varphi(x)$ for all $x \in \mathbb{R}^n$ $\mathrm{supp}(\psi_i \varphi) \subseteq U_i$

$$\Longrightarrow \langle T, \varphi \rangle = \left\langle T, \sum_{i=1}^{m} \psi_i \varphi \right\rangle = \sum_{i=1}^{m} \underbrace{\langle T, \psi_i \varphi \rangle}_{=0} = 0 \qquad \square$$

Distributions - Part 16

$T \in \mathcal{D}'(\mathbb{R}^n) \rightsquigarrow \operatorname{supp}(T) \subseteq \mathbb{R}^n$ closed set

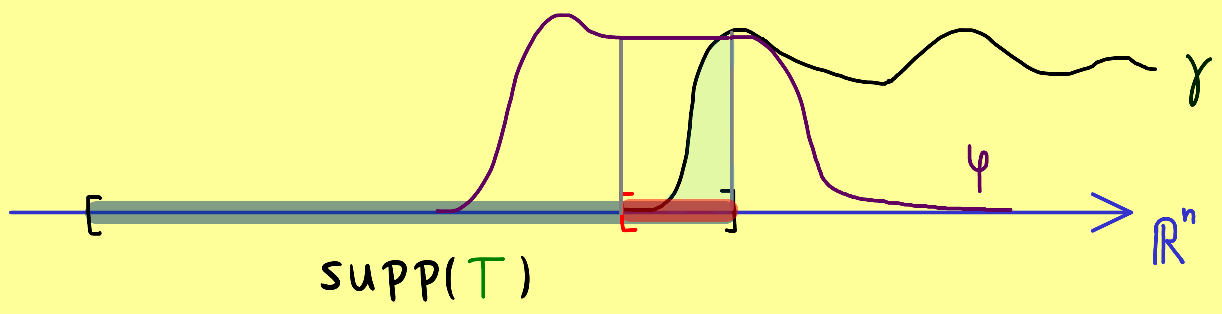


Definition: For $T \in \mathcal{D}'(\mathbb{R}^n)$, we define: $\mathcal{E}_T := \left\{ \gamma \in C^\infty(\mathbb{R}^n) \;\middle|\; \begin{array}{l} \operatorname{supp}(T) \cap \operatorname{supp}(\gamma) \\ \text{is compact in } \mathbb{R}^n \end{array} \right\}$

Extension for T to $\mathcal{E}_T$:

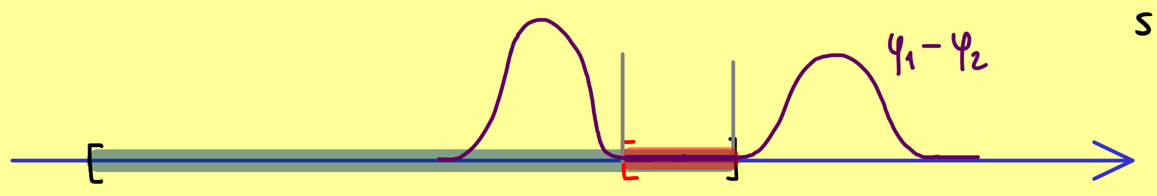
$$T(\gamma) = \langle T, \gamma \rangle := \langle T, \varphi \cdot \gamma \rangle \quad \text{for } \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ with}$$

$\varphi(x) = 1$, for all x in an open set that contains: $\operatorname{supp}(T) \cap \operatorname{supp}(\gamma)$



$\operatorname{supp}(\varphi \cdot \gamma) = \operatorname{supp}(\varphi) \cap \operatorname{supp}(\gamma)$

Is it well-defined? $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^n)$, $\varphi_1(x) = \varphi_2(x) = 1$, for all x in an open set that contains: $\operatorname{supp}(T) \cap \operatorname{supp}(\gamma)$



$$\langle T, \varphi_1 \gamma \rangle - \langle T, \varphi_2 \gamma \rangle = \langle T, (\varphi_1 - \varphi_2)\gamma \rangle = 0$$

Properties: $T : \mathcal{E}_T \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) satisfies:

(1) It's a linear functional on $\mathcal{E}_T$.

(2) Distributional derivatives: $\langle D^\alpha T, \gamma \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \gamma \rangle$

Common notation: For a distribution T where $\operatorname{supp}(T)$ is compact in $\mathbb{R}^n$:

$$T \in \mathcal{E}'(\mathbb{R}^n)$$

Distributions – Part 17

Convolution from part 13: $\ast : \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

defined by: $\langle \psi \ast T, \varphi \rangle = \langle T, \check{\psi} \ast \varphi \rangle$

where $\check{\psi}(x) := \psi(-x)$

Convolution (extended): $\ast : \mathcal{D}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

Definition: For $S \in \mathcal{E}'(\mathbb{R}^n)$, we define a new distribution:

$$\langle \check{S}, \varphi \rangle = \langle S, \check{\varphi} \rangle$$

Easy to show: $\left(\langle T_{\check{f}}, \varphi \rangle = \langle T_f, \check{\varphi} \rangle \right.$

We get: $\check{S} \in \mathcal{E}'(\mathbb{R}^n)$.

Proposition: For $\psi \in \mathcal{D}(\mathbb{R}^n)$, $S \in \mathcal{E}'(\mathbb{R}^n)$, we get:

$\psi \ast \check{S}$ is a regular distribution

and $\psi \ast \check{S} \in C^\infty(\mathbb{R}^n)$

and $\psi \ast \check{S} \in \mathcal{D}(\mathbb{R}^n)$ (supp($\psi \ast \check{S}$) compact)

Definition: The convolution $\ast : \mathcal{D}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

is given by $\langle T \ast S, \varphi \rangle := \langle T, \varphi \ast \check{S} \rangle$

Compatible to old definition: Choose regular distribution $T = T_\psi$ with $\psi \in \mathcal{D}(\mathbb{R}^n)$

$$\langle T_\psi \ast S, \varphi \rangle = \langle T_\psi, \overbrace{\varphi \ast \check{S}}^{= T_g} \rangle = \int_{\mathbb{R}^n} \psi(x)\, g(x) \, dx$$

$$= \langle \varphi \ast \check{S}, \psi \rangle = \langle \check{S}, \check{\varphi} \ast \psi \rangle$$

$$= \langle S, (\check{\varphi} \ast \psi)^\vee \rangle = \langle S, \check{\psi} \ast \varphi \rangle \quad ✓$$

Important property: $T \ast \delta = T$ for all $T \in \mathcal{D}'(\mathbb{R}^n)$

$\delta \ast S = S$ for all $S \in \mathcal{E}'(\mathbb{R}^n)$