The Bright Side of Mathematics

Distributions – part 5

Regular distributions

Proposition: $T: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) linear. Then:

T is a distribution $\iff \underset{\substack{K \subseteq \mathbb{R}^n \\ \text{compact}}}{\forall} \ \underset{m \in \mathbb{N}_0}{\exists} \ \underset{C>0}{\exists} \ \underset{\varphi \in \mathcal{D}(\mathbb{R}^n)}{\forall}$ $\quad \operatorname{supp}(\varphi) \subseteq K \implies |T(\varphi)| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$

Proof: ($\Leftarrow$) Let $\varphi_k, \varphi \in \mathcal{D}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ with $\varphi_k \xrightarrow[\mathcal{D}]{} \varphi$.

Then there is a $K \subseteq \mathbb{R}^n$ with $\operatorname{supp}(\varphi_k) \subseteq K$

and for all α we have $\|D^\alpha \varphi_k - D^\alpha \varphi\|_\infty \xrightarrow{k\to\infty} 0$.

$$|T(\varphi_k) - T(\varphi)| = |T(\varphi_k - \varphi)| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi_k - D^\alpha \varphi\|_\infty \xrightarrow{k\to\infty} 0$$

($\Rightarrow$) Proof by contraposition:

$$\underset{\substack{K \subseteq \mathbb{R}^n \\ \text{compact}}}{\exists} \ \underset{m \in \mathbb{N}_0}{\forall} \ \underset{C>0}{\forall} \ \underset{\varphi \in \mathcal{D}(\mathbb{R}^n)}{\exists} \quad \operatorname{supp}(\varphi) \subseteq K \text{ and } |T(\varphi)| > C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty$$

For $C = m = k \in \mathbb{N}$ take $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$ with $|T(\varphi_k)| > k \cdot \sum_{\underbrace{|\alpha|}_{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \leq k} \|D^\alpha \varphi_k\|_\infty$

Define: $\psi_k(x) := \dfrac{1}{|T(\varphi_k)|} \varphi_k(x)$. Then: $\psi_k \xrightarrow[\mathcal{D}]{} 0$

But: $|T(\psi_k)| = \dfrac{1}{|T(\varphi_k)|} \cdot |T(\varphi_k)| = 1 \ \overset{k\to\infty}{\not\longrightarrow} \ 0$ $\square$

Definition: $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) is called <u>locally integrable</u>

if for all compact $K \subseteq \mathbb{R}^n$: $\displaystyle\int_K |f(x)|\, dx < \infty$

Then we write: $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$.

For example: $f: \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2 \implies f \in \mathcal{L}^1_{loc}(\mathbb{R})$

For $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, define $T_f \in \mathcal{D}'(\mathbb{R}^n)$ by $T_f(\varphi) = \displaystyle\int_{\mathbb{R}^n} f(x)\, \varphi(x)\, dx$

$$\boxed{|T(\varphi)| \leq C \cdot \sum_{|\alpha| \leq m} \|D^\alpha \varphi\|_\infty} \ \checkmark$$

$$|T_f(\varphi)| \leq \int_{\operatorname{supp}(\varphi)} |f(x)| \cdot \underbrace{|\varphi(x)|}_{\leq \|\varphi\|_\infty} dx \leq \int_K |f(x)|\, dx \cdot \|\varphi\|_\infty$$

$K \leftarrow$ compact set $K \supseteq \operatorname{supp}(\varphi)$

Definition: $T \in \mathcal{D}'(\mathbb{R}^n)$ is called <u>regular</u> if there is a locally integrable function f such that $T = T_f$.