



Complex Analysis - Part 35

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6} dx = ?$$

$$\equiv \lim_{R \rightarrow \infty} \int_{-R}^{-R} f(x) dx$$

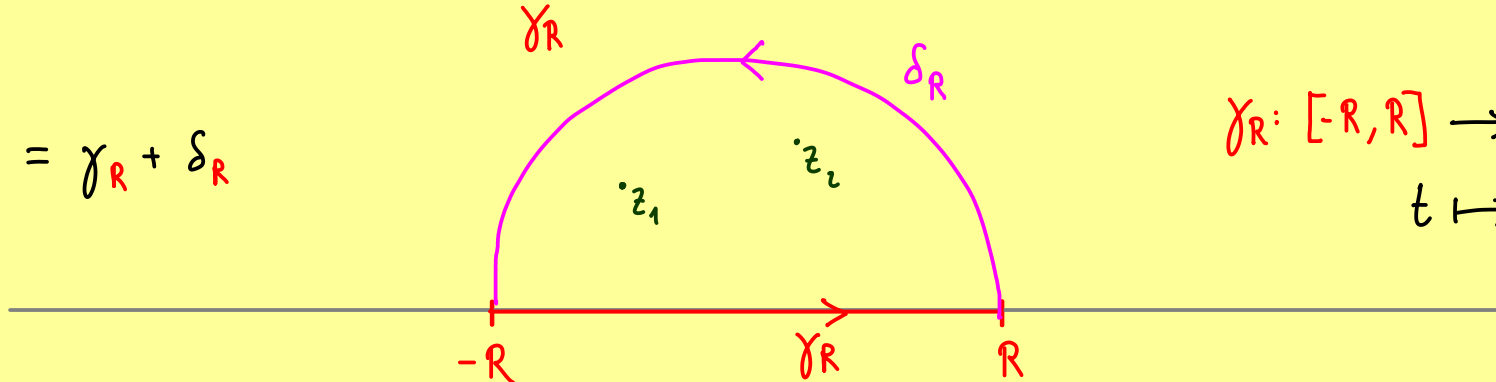
$$\text{where } f(x) = \frac{x^4}{1+x^6}$$

complex contour integral:

$$\int_{\Gamma_R} f(z) dz$$

$$\text{where } f(z) = \frac{z^4}{1+z^6}$$

$$\Gamma_R = \gamma_R + \delta_R$$



$$\gamma_R: [-R, R] \rightarrow \mathbb{C}$$

$$t \mapsto t$$

residue theorem:

$$\oint_{\Gamma_R} f(z) dz = 2\pi i \sum_{j=1}^n \text{Res}(f, z_j)$$

$$\int_{\gamma_R} f(z) dz + \int_{\delta_R} f(z) dz \quad \text{where} \quad \left| \int_{\delta_R} f(z) dz \right| \leq \max_{z \in \text{Ran}(\delta_R)} |f(z)| \cdot \text{length}(\delta_R)$$

$$\frac{(R e^{it})^4}{1+(R e^{it})^6}$$

$$\leq C \cdot \frac{1}{R^2} \cdot \pi \cdot R \xrightarrow{R \rightarrow \infty} 0$$

Hence:

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6} dx = \lim_{R \rightarrow \infty} \int_{\gamma_R} f(z) dz = \lim_{R \rightarrow \infty} \oint_{\Gamma_R} f(z) dz$$

$$= 2\pi i \sum_{\text{Im}(z) > 0} \text{Res}(f, z)$$

$$\text{poles: } 1+x^6 = 0 \Rightarrow z_1 = e^{i\frac{\pi}{6}}, z_2 = e^{3i\frac{\pi}{6}}, z_3 = e^{5i\frac{\pi}{6}}$$

$$\text{formula for simple poles: } \text{Res}\left(\frac{h}{g}, z\right) = \frac{h(z)}{g'(z)}$$

$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} \frac{x^4}{1+x^6} dx &= 2\pi i \cdot \left(\frac{1}{6} e^{-i\frac{\pi}{6}} + \frac{1}{6} e^{-3i\frac{\pi}{6}} + \frac{1}{6} e^{-5i\frac{\pi}{6}} \right) \\ &= \frac{1}{3} \pi i \left(\underbrace{i \sin\left(-\frac{\pi}{6}\right)}_{-\sin\left(\frac{\pi}{6}\right)} + \underbrace{i \sin\left(-\frac{3\pi}{6}\right)}_{=-1} + \underbrace{i \sin\left(-\frac{5\pi}{6}\right)}_{-\sin\left(\frac{\pi}{6}\right)} \right) \\ &= \frac{1}{3} \pi \left(1 + 2 \sin\left(\frac{\pi}{6}\right) \right) \end{aligned}$$