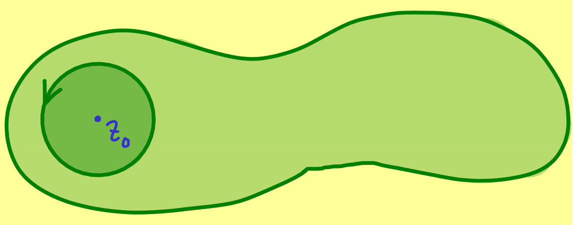
Complex Analysis – Part 29

Cauchy's inequalities: $f : D \longrightarrow \mathbb{C}$ holomorphic, $\overline{B_r(z_0)} \subseteq D$.

Then: $\left| f^{(n)}(z_0) \right| \leq \dfrac{n!}{r^n} \cdot \sup\limits_{z \in \partial B_r(z_0)} |f(z)|$



Proof: $$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right|$$

parametrized curve: $r \cdot e^{it} + z_0$, $t \in [0, 2\pi]$

$$= \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(r \cdot e^{it} + z_0)}{(r \cdot e^{it})^{n+1}} \cdot r i e^{it} \, dt \right|$$

$$= \left| \frac{n!}{2\pi} \cdot \frac{1}{r^n} \int_0^{2\pi} f(r \cdot e^{it} + z_0) \, e^{it(-n)} \, dt \right|$$

$$\leq \frac{n!}{2\pi} \cdot \frac{1}{r^n} \int_0^{2\pi} \underbrace{\left| f(r \cdot e^{it} + z_0) \right|}_{\leq \sup\limits_{z \in \partial B_r(z_0)} |f(z)|} dt \leq \frac{n!}{\cancel{2\pi}} \cdot \frac{1}{r^n} \cdot \cancel{2\pi} \cdot \sup_{z \in \partial B_r(z_0)} |f(z)|$$

$\square$

Application: $f : \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic and bounded $\left(\sup\limits_{z \in \mathbb{C}} |f(z)| = c \right)$

(Liouville's theorem)

$\Longrightarrow \left| f'(z_0) \right| \leq \dfrac{1!}{r^1} c$ for all $r > 0$, $z_0 \in \mathbb{C}$

$\Longrightarrow f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$

$\Longrightarrow f : \mathbb{C} \longrightarrow \mathbb{C}$ is constant

($\sin : \mathbb{C} \longrightarrow \mathbb{C}$ not bounded)