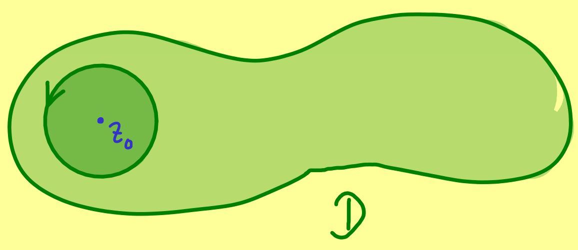
The Bright Side of Mathematics

Complex Analysis – Part 28

Fact: $f : D \to \mathbb{C}$ holomorphic. Then:



(a) $f^{(n)}(z)$ exists for all $z \in D$, $n \in \mathbb{N}$

(b) $$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

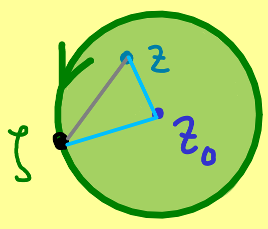
for all $z \in B_r(z_0)$.

(c) In $B_r(z_0)$, f is a power series:

$$f(z) = \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k \quad \text{for} \quad a_k = \frac{1}{k!} \cdot f^{(k)}(z_0)$$

Proof: $$2\pi i \cdot f(z) = \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Cauchy's integral formula



$$= \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} d\zeta$$

$$= \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} \cdot \frac{1}{1 - \underbrace{\frac{z - z_0}{\zeta - z_0}}_{=:q}} d\zeta \quad , \quad |q| = \frac{\overbrace{|z - z_0|}^{< r}}{\underbrace{|\zeta - z_0|}_{= r}} < 1$$

geometric series

$$= \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} \cdot \sum_{k=0}^{\infty} q^k \, d\zeta$$

uniform convergence

$$= \sum_{k=0}^{\infty} \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} \cdot \left(\frac{z - z_0}{\zeta - z_0}\right)^k d\zeta$$

$$= \sum_{k=0}^{\infty} \tilde{a}_k \cdot (z - z_0)^k \quad \text{for} \quad \tilde{a}_k = \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$