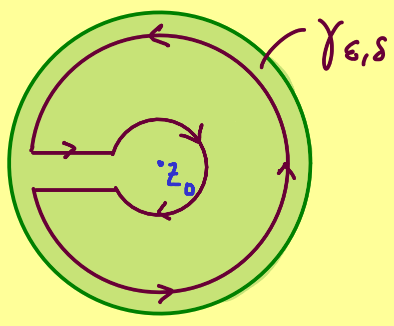
The Bright Side of Mathematics

Complex Analysis – Part 26

⇝ Cauchy's theorem applicable

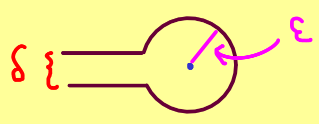
⇝ keyhole contour

Assume: $g: B_r(z_0)\setminus\{z_0\} \longrightarrow \mathbb{C}$ holomorphic

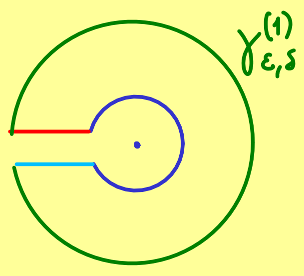


Cauchy's theorem

$$\oint_{\gamma_{\varepsilon,\delta}} g(z)\,dz = 0$$



Split it up:

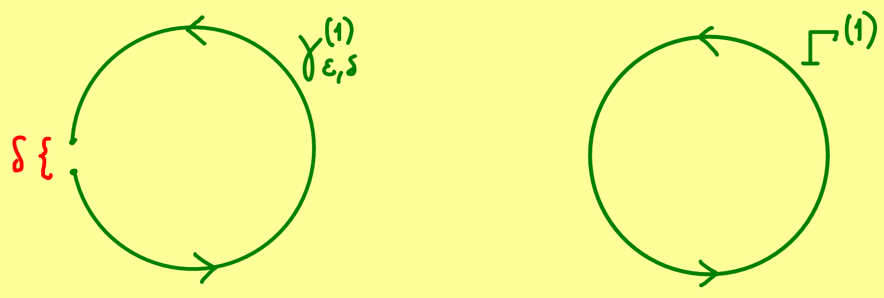


$$\gamma_{\varepsilon,\delta} = \gamma^{(1)}_{\varepsilon,\delta} + \gamma^{(2)}_{\varepsilon,\delta} + \gamma^{(3)}_{\varepsilon,\delta} + \gamma^{(4)}_{\varepsilon,\delta}$$

$$\Longrightarrow \int_{\gamma^{(1)}_{\varepsilon,\delta}} g(z)\,dz + \int_{\gamma^{(2)}_{\varepsilon,\delta}} g(z)\,dz + \int_{\gamma^{(4)}_{\varepsilon,\delta}} g(z)\,dz + \int_{\gamma^{(3)}_{\varepsilon,\delta}} g(z)\,dz = 0$$

What happens for $\delta \to 0$?

(1)



$$\left| \int_{\gamma^{(1)}_{\varepsilon,\delta}} g(z)\,dz - \int_{\Gamma^{(1)}} g(z)\,dz \right| = \left| \int_{(} g(z)\,dz \right| \leq \max_{z \in (} |g(z)| \cdot \text{length}(\,(\,) \xrightarrow{\delta \to 0} 0$$

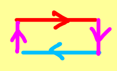
(3)

(2) (4)

$\gamma^{(2)}_{\varepsilon,\delta}$

$\gamma^{(4)}_{\varepsilon,\delta}$

⇝



Cauchy's theorem

$$\Longrightarrow \oint g(z)\,dz = 0$$

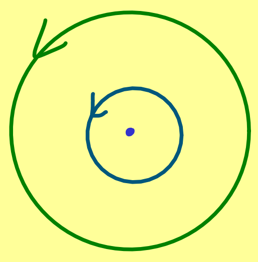
$$\overset{\delta \to 0}{\Longrightarrow} \int_{\rightarrow} g(z)\,dz + \int_{\leftarrow} g(z)\,dz = 0$$

In summary: For $\delta \to 0$:

$$\int_{\Gamma^{(1)}} g(z)\,dz + \int_{\Gamma^{(3)}_{\varepsilon}} g(z)\,dz = 0$$

Result:

$$\int_{\Gamma^{(1)}} g(z)\,dz = \int_{\Gamma^{(3)-}_{\varepsilon}} g(z)\,dz$$



same integral value