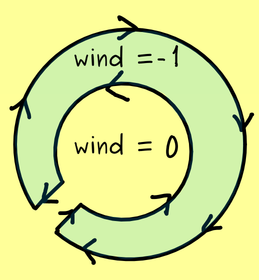
BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Complex Analysis – Part 25

winding number: $\operatorname{wind}(\gamma, z_0) := \frac{1}{2\pi i} \oint_\gamma \frac{1}{z - z_0}\, dz$



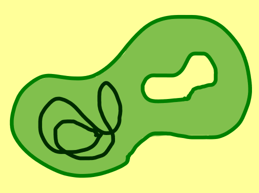
Definition: For $\gamma : [a,b] \to \mathbb{C}$ closed:

$$\operatorname{Ext}(\gamma) := \{ z_0 \in \mathbb{C} \setminus \operatorname{Ran}(\gamma) \mid \operatorname{wind}(\gamma, z_0) = 0 \}$$

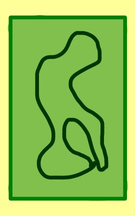
$$\operatorname{Int}(\gamma) := \{ z_0 \in \mathbb{C} \setminus \operatorname{Ran}(\gamma) \mid \operatorname{wind}(\gamma, z_0) \neq 0 \}$$

Extending Cauchy's theorem:

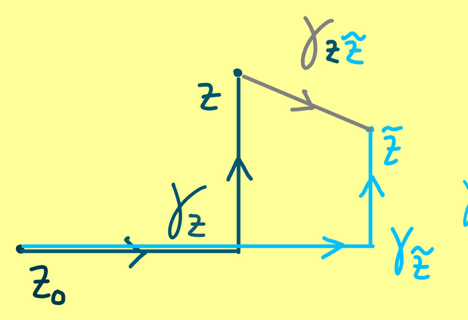
$f : D \to \mathbb{C}$ holomorphic, γ closed, $\operatorname{Int}(\gamma) \cup \operatorname{Ran}(\gamma) \subseteq D$



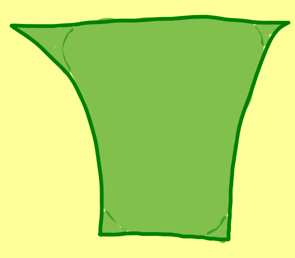
$D =$ disc  $\overset{\text{part 23}}{\Longrightarrow} \oint_\gamma f(z)\, dz = 0$

$D =$ rectangle  $\overset{\text{same proof part 23}}{\Longrightarrow} \oint_\gamma f(z)\, dz = 0$

proof needed: $F(z) := \int_{\gamma_z} f(\zeta)\, d\zeta$



$\oint_{\gamma_{\tilde z} + \gamma_z + \gamma_{z\tilde z}} f(z)\, dz \overset{\text{Goursat}}{=} 0$

works also: $D =$  , $D =$ 

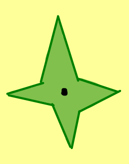
Cauchy's theorem (general version):

$f : D \to \mathbb{C}$ holomorphic, γ closed, $\operatorname{Int}(\gamma) \cup \operatorname{Ran}(\gamma) \subseteq D$

$$\Longrightarrow \oint_\gamma f(z)\, dz = 0$$

Cauchy's theorem (for some domains):

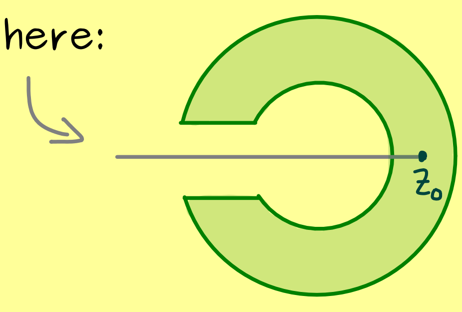
$f : D \to \mathbb{C}$ holomorphic, $\gamma : [a,b] \to D$ closed curve,

If D convex  or $D =$  or D star domains  $\Longrightarrow \oint_\gamma f(z)\, dz = 0$

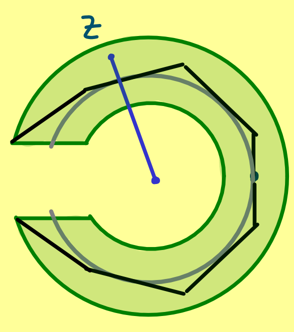
Appendix:

Proof from part 23 can be transformed to a proof for domain $D =$ 

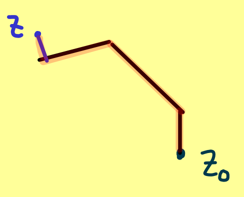
Just fix point z_0 here:



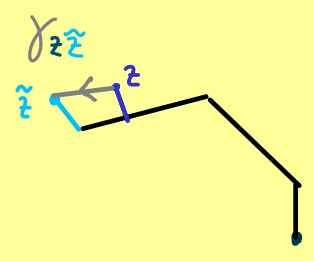
Then for every z, define the path γ_z in the following way:



So it's a well-defined polygon path:



Hence, we have $\oint_{\gamma_{\tilde z} + \bar\gamma_z + \gamma_{z\tilde z}} f(z)\, dz \overset{\text{Goursat}}{=} 0$ (*) if $\tilde z$ and z are close enough.



So for $F(z) := \int_{\gamma_z} f(\zeta)\, d\zeta$, we also get:

$$\left| \frac{F(\tilde z) - F(z)}{\tilde z - z} - f(z) \right| = \frac{1}{|\tilde z - z|} \cdot \left| \int_{\gamma_{\tilde z}} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta - f(z)(\tilde z - z) \right|$$

$$\overset{(*)}{=} \frac{1}{|\tilde z - z|} \cdot \left| \int_{\gamma_{z\tilde z}} \big(f(\zeta) - f(z)\big)\, d\zeta \right|$$

$$\leq \frac{1}{\cancel{|\tilde z - z|}} \max_{\zeta \in \operatorname{Ran}(\gamma_{z\tilde z})} \big| f(\zeta) - f(z) \big| \cdot \cancel{\operatorname{length}}(\gamma_{z\tilde z}) \overset{\tilde z \to z}{\longrightarrow} 0$$

$\Longrightarrow f$ has an antiderivative on D $\Longrightarrow \oint_\gamma f(z)\, dz = 0$ for each closed curve γ in D