BECOME A MEMBER

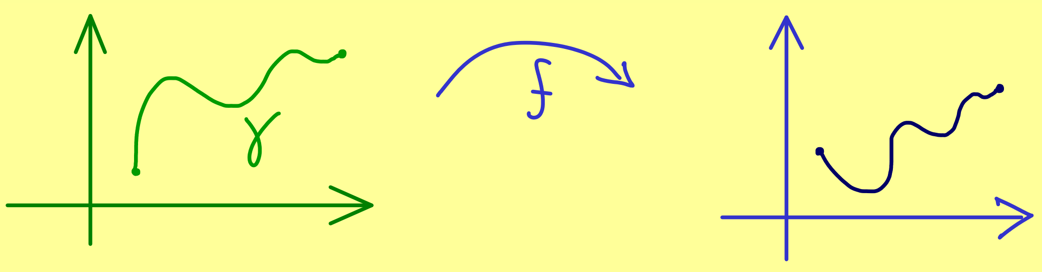
ON STEADY

The Bright Side of
Mathematics

Complex Analysis – Part 17

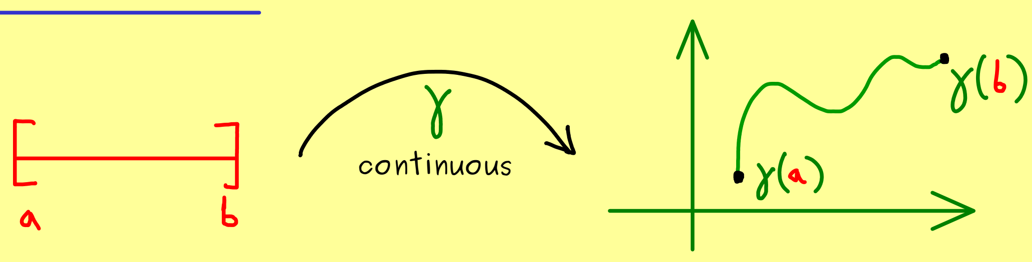
Complex integration: $f: \mathbb{C} \to \mathbb{C}$

↳ curve integral, line integral, contour integral



$$\int_\gamma f(z)\, dz$$

Complex integration on real intervals:



For a continuous map $\gamma: [a,b] \to \mathbb{C}$, we define:

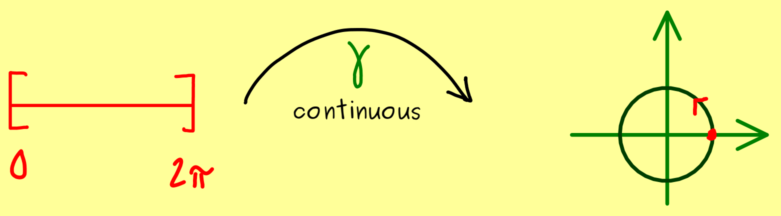
$$\int_a^b \gamma(t)\, dt := \int_a^b \mathrm{Re}(\gamma(t))\, dt + i \cdot \int_a^b \mathrm{Im}(\gamma(t))\, dt$$

ordinary Riemann integrals in $\mathbb{R}$

Important property: Let $\gamma: [a,b] \to \mathbb{C}$ be continuous. Then:

$$\left| \int_a^b \gamma(t)\, dt \right| \leq \int_a^b |\gamma(t)|\, dt$$

Example: $\gamma: [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$



$$\int_a^b e^{it}\, dt = \int_a^b \cos(t)\, dt + i \cdot \int_a^b \sin(t)\, dt$$

$$= \sin(t)\Big|_a^b + i \cdot (-\cos(t))\Big|_a^b = -i \cdot \cos(t) + \sin(t) \Big|_a^b$$

$$= \frac{1}{i}\left(\cos(t) + i \cdot \sin(t)\right)\Big|_a^b = \frac{1}{i} e^{it} \Big|_a^b$$

Proof: of $\boxed{\left|\int_a^b \gamma(t)\, dt\right| \leq \int_a^b |\gamma(t)|\, dt}$

Assume $0 \neq \underbrace{\int_a^b \gamma(t)\, dt}_{=: w} \in \mathbb{C}$. Define: $c := \frac{w}{|w|}$. Then:

$$\int_a^b \mathrm{Re}(c^{-1} \cdot \gamma(t))\, dt = \int_a^b c^{-1} \cdot \gamma(t)\, dt = c^{-1} \int_a^b \gamma(t)\, dt = |w| \in \mathbb{R}$$

We know: $\left|\mathrm{Re}(c^{-1} \cdot \gamma(t))\right| \leq \left|c^{-1} \cdot \gamma(t)\right| = \underbrace{|c^{-1}|}_{=1} \cdot |\gamma(t)|$

$$\Rightarrow \int_a^b \left|\mathrm{Re}(c^{-1} \cdot \gamma(t))\right| dt \leq \int_a^b |\gamma(t)|\, dt$$

$$\rotatebox{45}{$\leq$}$$

$$\left| \int_a^b \gamma(t)\, dt \right| = \left| \int_a^b \mathrm{Re}(c^{-1} \cdot \gamma(t))\, dt \right|$$