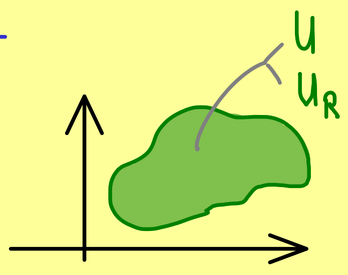
Complex Analysis – Part 7



<u>Theorem:</u> $U \subseteq \mathbb{C}$ open.

$f: U \to \mathbb{C}$ is holomorphic

$\Longleftrightarrow$ Real part of f as a function on $U_\mathbb{R} \subseteq \mathbb{R}^2$

$u: U_\mathbb{R} \to \mathbb{R}$

and imaginary part of f as a function on $U_\mathbb{R} \subseteq \mathbb{R}^2$

$v: U_\mathbb{R} \to \mathbb{R}$

fulfil:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
at all points $(x,y) \in U_\mathbb{R}$

<u>Examples:</u>

(a) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z \Rightarrow f(x+iy) = \underbrace{x}_{u(x,y)} + i\underbrace{y}_{v(x,y)}$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 0$$
$$\frac{\partial v}{\partial y} = 1 \qquad -\frac{\partial v}{\partial x} = 0 \quad \Rightarrow f \text{ is holomorphic}$$

(b) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z} \Rightarrow f(x+iy) = \underbrace{x}_{u(x,y)} + i\underbrace{(-y)}_{v(x,y)}$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \quad \Rightarrow f \text{ is not holomorphic}$$

(c) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 + iz \Rightarrow f(x+iy) = (x+iy)^2 + i(x+iy)$

$$= x^2 + i2xy - y^2 + ix - y$$
$$= \underbrace{(x^2 - y^2 - y)}_{u(x,y)} + i\underbrace{(2xy + x)}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y - 1$$
$$\frac{\partial v}{\partial y} = 2x \qquad -\frac{\partial v}{\partial x} = -(2y+1)$$

$\Rightarrow f$ is holomorphic