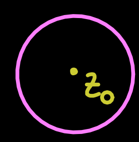
Complex Analysis – Part 30

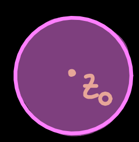
$f : \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic

with known values $\{ (z, f(z)) \mid z \in \partial B_r(z_0) \}$



$g : \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic with same values on $\partial B_r(z_0)$

$\overset{\text{Cauchy's integral formula}}{\Longrightarrow}$ $f = g$ on $B_r(z_0)$



Identity theorem: $f, g : D \longrightarrow \mathbb{C}$ holomorphic, $D \subseteq \mathbb{C}$ open domain (connected).

Then: $\{ z \in D \mid f(z) = g(z) \}$ has an accumulation point in D

$$\Longleftrightarrow$$

$$f = g$$

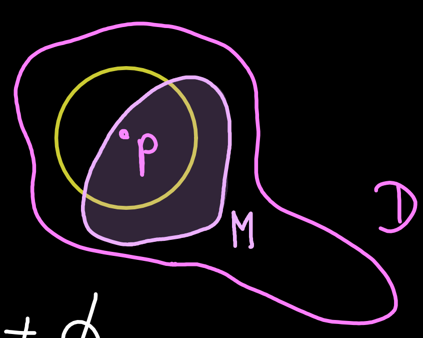
$$\Longleftrightarrow$$

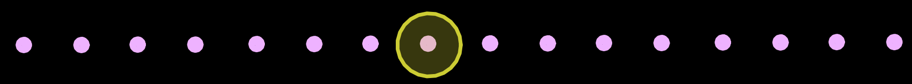
There is $c \in D$ with $f^{(n)}(c) = g^{(n)}(c)$ for all $n = 0, 1, 2, \dots$

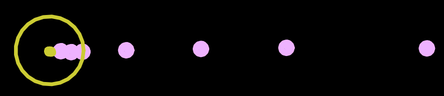
What is an accumulation point?

$p \in D$ is called an accumulation point of the set $M \subseteq D$

if for all open set U with $p \in U$: $U \setminus \{p\} \cap M \neq \emptyset$



$M = \mathbb{N}$  no accumulation point

$M = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$  0 is accumulation point

Proof idea: $h := f - g$ holomorphic. Show the equivalence of:

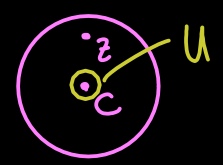
(1) $M = \{ z \in D \mid h(z) = 0 \}$ has an accumulation point in D

(2) $h = 0$

(3) There is $c \in D$ with $h^{(n)}(c) = 0$ for all $n = 0, 1, 2, \ldots$

(1) $\Rightarrow$ (3) (Contraposition: $\neg(3) \Rightarrow \neg(1)$)

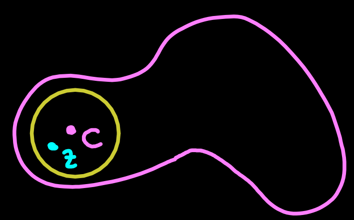
For each $c \in D$ there is a minimal m with $h^{(m)}(c) \neq 0$

and $h(z) = \sum_{k=m}^{\infty} \underbrace{\frac{h^{(k)}(c)}{k!}}_{a_k} (z-c)^k = \underset{\neq 0}{a_m} \cdot (z-c)^m + \cdots$ 

$\Rightarrow h(z) \neq 0$ for $z \in U \setminus \{c\}$

$\Rightarrow U \setminus \{c\} \cap M = \emptyset$

(3) $\Rightarrow$ (2) $A_k := \{ z \in D \mid h^{(k)}(z) = 0 \}$ closed $\Rightarrow \underset{\neq \emptyset \ (3)}{A} := \bigcap_{k=0}^{\infty} A_k$ closed

A is also open: $c \in A$ 

$\overset{D \text{ connected}}{\Longrightarrow} A = D \Rightarrow h = 0$

(2) $\Rightarrow$ (1) ✓

$\square$