BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Complex Analysis – Part 30

$f: \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic

with known values $\{(z, f(z)) \mid z \in \partial B_r(z_0)\}$

$g: \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic with same values on $\partial B_r(z_0)$

Cauchy's integral formula

$\Longrightarrow$ $f = g$ on $B_r(z_0)$

Identity theorem: $f, g: D \longrightarrow \mathbb{C}$ holomorphic, $D \subseteq \mathbb{C}$ open domain (connected).

Then: $\{z \in D \mid f(z) = g(z)\}$ has an accumulation point in D

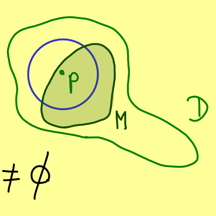
$\Longleftrightarrow$

$f = g$

$\Longleftrightarrow$

There is $c \in D$ with $f^{(n)}(c) = g^{(n)}(c)$ for all $n = 0, 1, 2, \dots$

What is an accumulation point?



$p \in D$ is called an accumulation point of the set $M \subseteq D$

if for all open set U with $p \in U$: $U \setminus \{p\} \cap M \neq \emptyset$

$M = \mathbb{N}$ $\quad \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$ no accumulation point

$M = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ $\quad$ 0 is accumulation point

Proof idea: $h := f - g$ holomorphic. Show the equivalence of:

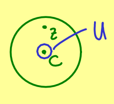
(1) $M = \{z \in D \mid h(z) = 0\}$ has an accumulation point in D

(2) $h = 0$

(3) There is $c \in D$ with $h^{(n)}(c) = 0$ for all $n = 0, 1, 2, \dots$

(1) ⇒ (3) (Contraposition: $\neg(3) \Rightarrow \neg(1)$)

For each $c \in D$ there is a minimal m with $h^{(m)}(c) \neq 0$



and $$h(z) = \sum_{k=m}^{\infty} \underbrace{\frac{h^{(k)}(c)}{k!}}_{a_k} (z-c)^k = \underbrace{a_m}_{\neq 0} \cdot (z-c)^m + \dots$$

$\Longrightarrow h(z) \neq 0$ for $z \in U \setminus \{c\}$

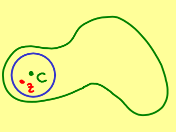
$\Longrightarrow U \setminus \{c\} \cap M = \emptyset$

(3) ⇒ (2)

$A_k := \{z \in D \mid h^{(k)}(z) = 0\}$ closed $\Longrightarrow A := \bigcap_{k=0}^{\infty} A_k$ closed

$A \neq \emptyset$ (3)

A is also open: $c \in A$



$\overset{D \text{ connected}}{\Longrightarrow} A = D \Rightarrow h = 0$

(2) ⇒ (1) ✓

□