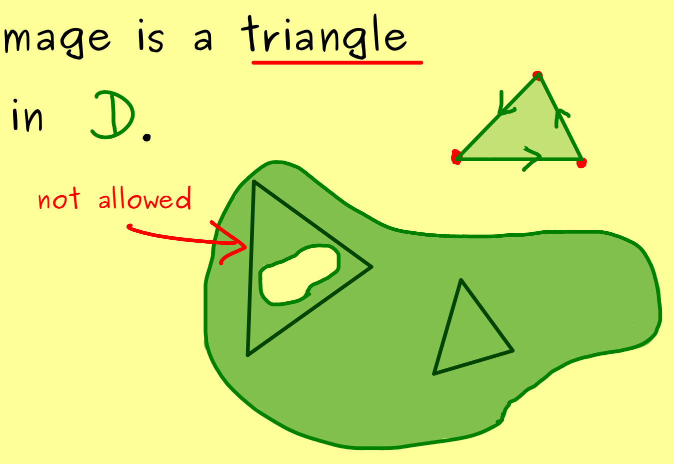
The Bright Side of Mathematics

Complex Analysis – Part 22

Goursat's theorem: $f: D \to \mathbb{C}$ holomorphic,

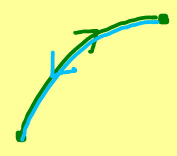
$\gamma: [a,b] \to D$ closed curve where the image is a triangle and the inner part lies in D.



Then: $$\oint_\gamma f(z)\,dz = 0$$

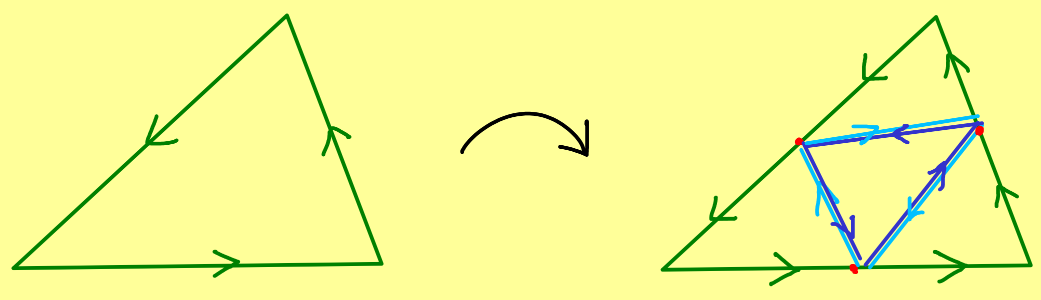
Proof:

Basic idea:

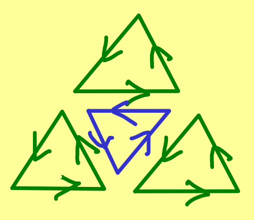


$$0 = \int_{\nearrow} + \int_{\swarrow} = \oint_{\nearrow + \swarrow}$$

Decompose triangle:



$$\oint_\gamma f(z)\,dz = \oint_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)\,dz$$



$\gamma_1+\gamma_2+\gamma_3+\gamma_4$

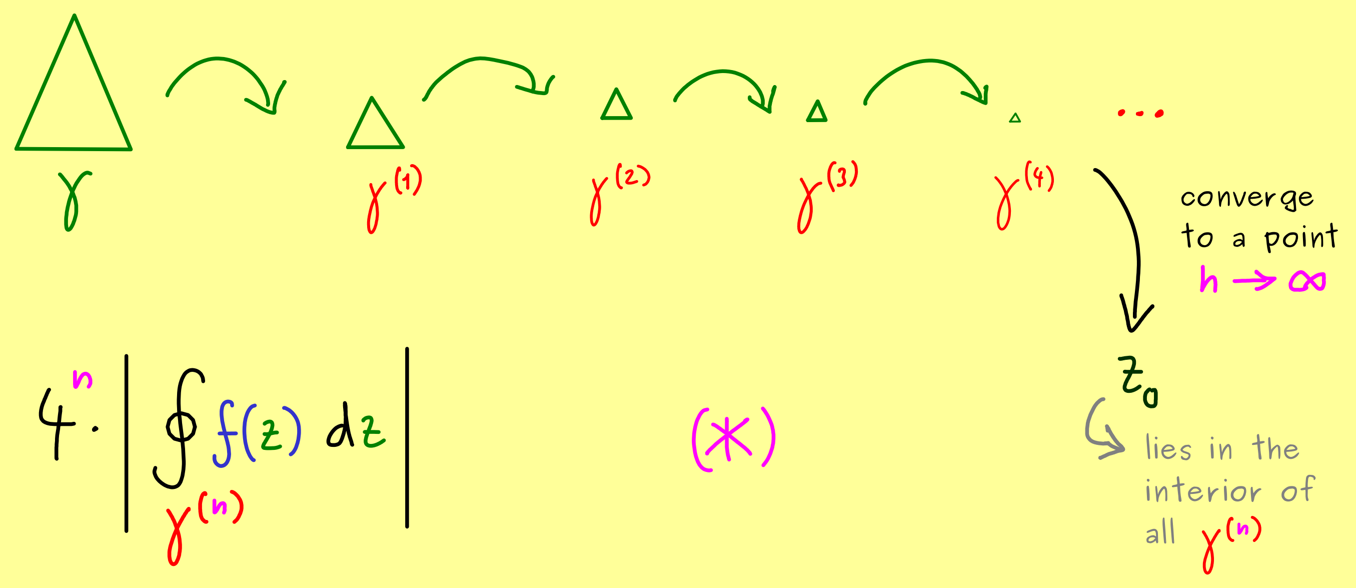
$$= \oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz + \oint_{\gamma_3} f(z)\,dz + \oint_{\gamma_4} f(z)\,dz$$

$$\left|\oint_\gamma f(z)\,dz\right| \leq \left|\oint_{\gamma_j} f(z)\,dz\right| + \left|\oint_{\gamma_j} f(z)\,dz\right| + \left|\oint_{\gamma_j} f(z)\,dz\right| + \left|\oint_{\gamma_j} f(z)\,dz\right|$$

$$= 4 \cdot \left|\oint_{\gamma^{(1)}} f(z)\,dz\right|$$

γ_j represents maximal value $= \gamma^{(1)}$

Repeat n times:



$$\left|\oint_\gamma f(z)\,dz\right| \leq 4^n \cdot \left|\oint_{\gamma^{(n)}} f(z)\,dz\right| \qquad (*)$$

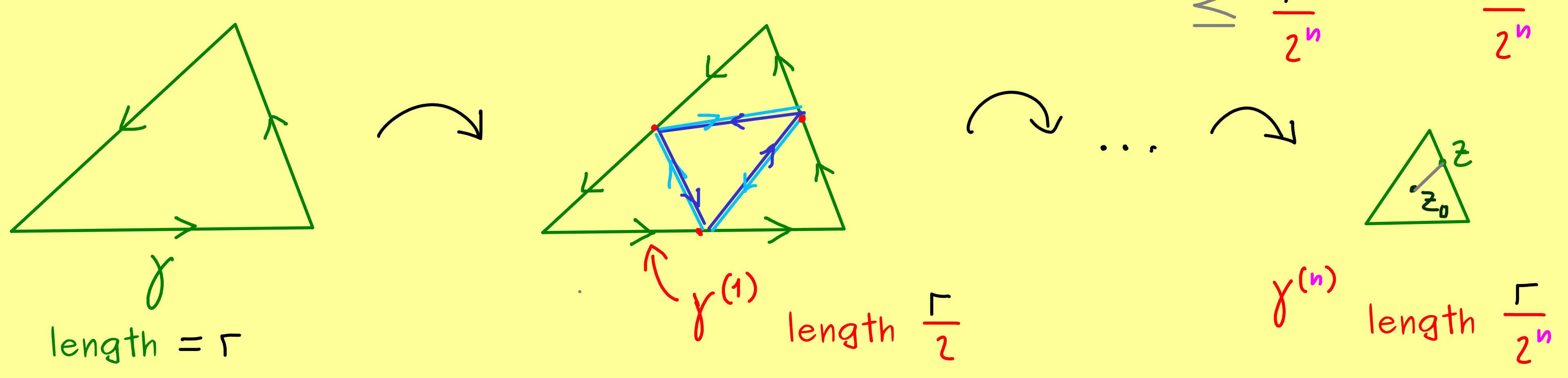
Complex differentiability at z_0:

$$f(z) = \underbrace{f(z_0) + f'(z_0)\cdot(z-z_0)}_{\text{has antiderivative } \Rightarrow \oint = 0} + \varphi(z) \qquad \text{where } \frac{\varphi(z)}{z-z_0} \xrightarrow{z \to z_0} 0$$

$$\varphi(z) = \psi(z)(z-z_0) \quad \text{with } \psi(z) \xrightarrow{z \to z_0} 0$$

$$\left|\oint_{\gamma^{(n)}} f(z)\,dz\right| = \left|\oint_{\gamma^{(n)}} \varphi(z)\,dz\right| \leq \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\varphi(z)| \cdot \mathrm{length}(\gamma^{(n)})$$

$$\leq \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\psi(z)| \cdot \underbrace{\max_{z \in \mathrm{Ran}(\gamma^{(n)})} |z-z_0|}_{\leq \frac{r}{2^n}} \cdot \underbrace{\mathrm{length}(\gamma^{(n)})}_{\frac{r}{2^n}}$$



$$\left|\oint_\gamma f(z)\,dz\right| \overset{(*)}{\leq} 4^n \cdot \left|\oint_{\gamma^{(n)}} f(z)\,dz\right| \leq r^2 \cdot \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\psi(z)| \xrightarrow{n \to \infty} 0$$

$\square$