

# Complex Analysis - Part 8

$f: U \rightarrow \mathbb{C}$  holomorphic

$$\frac{\partial f}{\partial z}(z_0) \stackrel{?}{=} f'(z_0) \quad \text{Wirtinger derivatives} \quad \frac{\partial f}{\partial \bar{z}}(z_0) \stackrel{?}{=} 0$$

$$\begin{aligned} f'(x+iy) &= \underbrace{a}_{\frac{\partial u}{\partial x}(x,y)} + i \underbrace{b}_{\frac{\partial v}{\partial x}(x,y)} \quad \text{for } f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \\ &= \frac{1}{2} \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + \underbrace{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}}_{\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}} \right) \quad \text{and map } \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \underbrace{f(x+iy)}_{u(x,y) + i v(x,y)} \\ &= \frac{1}{2} \left( \underbrace{\frac{\partial}{\partial x}(u+iv)}_{\frac{\partial f}{\partial x}} - i \underbrace{\frac{\partial}{\partial y}(u+iv)}_{\frac{\partial f}{\partial y}} \right) \end{aligned}$$

Definition:  $\frac{\partial}{\partial z} := \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad , \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \cdot \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$

Example:  $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i \cdot 2 \cdot x \cdot y \Rightarrow \frac{\partial f}{\partial x} = 2 \cdot x + i 2y = 2 \cdot z$   
 $\frac{\partial f}{\partial y} = -2y + i 2x = 2 \cdot i z$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} (2z + i \cdot 2iz) = 0 \quad , \quad \frac{\partial f}{\partial z} = \frac{1}{2} (2z - i \cdot 2iz) = 2 \cdot z$$

Fact:  $f: U \rightarrow \mathbb{C}$  holomorphic  $\iff \frac{\partial f}{\partial \bar{z}} = 0$  at all points in  $U$

In this case:  $f' = \frac{\partial f}{\partial z}$