Complex Analysis – Part 8

$f: U \longrightarrow \mathbb{C}$ holomorphic

Wirtinger derivatives

$$\frac{\partial f}{\partial z}(z_0) \overset{?}{=} f'(z_0) \qquad\qquad \frac{\partial f}{\partial \bar{z}}(z_0) \overset{?}{=} 0$$

$$f'(x+iy) = \underbrace{a}_{\frac{\partial u}{\partial x}(x,y)} + i\underbrace{b}_{\frac{\partial v}{\partial x}(x,y)} \qquad \text{for} \qquad f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

and map $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto f(x+iy) = u(x,y) + i\,v(x,y)$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial v}{\partial y}} + i\underbrace{\frac{\partial v}{\partial x}}_{-\frac{\partial u}{\partial y}} \right)$$

$$= \frac{1}{2}\left(\underbrace{\frac{\partial}{\partial x}(u+iv)}_{\frac{\partial f}{\partial x}} - i\underbrace{\frac{\partial}{\partial y}(u+iv)}_{\frac{\partial f}{\partial y}} \right)$$

Definition:

$$\frac{\partial}{\partial z} := \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad , \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Example:

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\cdot 2\cdot x\cdot y \quad \Rightarrow \quad \frac{\partial f}{\partial x} = 2x + i2y = 2\cdot z$$

$$\frac{\partial f}{\partial y} = -2y + i2x = 2\cdot i\, z$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(2z + i\cdot 2iz\right) = 0 \quad , \quad \frac{\partial f}{\partial z} = \frac{1}{2}\left(2z - i\cdot 2iz\right) = 2\cdot z$$

Fact: $f: U \longrightarrow \mathbb{C}$ holomorphic $\Longleftrightarrow$ $\frac{\partial f}{\partial \bar{z}} = 0$ at all points in U

In this case: $f' = \frac{\partial f}{\partial z}$