# Complex Analysis – Part 6

(1) $f : \mathbb{C} \to \mathbb{C}$ is (complex) differentiable at $z_0 \in \mathbb{C}$ if

there is $f'(z_0) \in \mathbb{C}$ and a function $\varphi : \mathbb{C} \to \mathbb{C}$ with:

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + \varphi(z) \quad \text{where} \quad \frac{\varphi(z)}{z - z_0} \xrightarrow{z \to z_0} 0$$

(2) $f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$ is called (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ if

there is a matrix $J \in \mathbb{R}^{2\times 2}$ and a map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with:

$$f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f_{\mathbb{R}}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + J\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + \phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \quad \text{where} \quad \frac{\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} 0$$

Question: In which cases does a matrix-vector multiplication represent a multiplication of complex numbers?

Let's check:

$$\underbrace{w}_{(a+ib)} \cdot \underbrace{z}_{(x+iy)} = (a \cdot x - by) + i \cdot (bx + ay)$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \cdot x - by \\ bx + ay \end{pmatrix}$$

Theorem: $f : \mathbb{C} \to \mathbb{C}$ is (complex) differentiable at $z_0 = x_0 + iy_0 \in \mathbb{C}$

$\iff$ $f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$ is (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$

and the Jacobian matrix at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ has the form: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$\iff$ For $f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ the Cauchy-Riemann equations are satisfied:

two maps: $\mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at point } (x_0, y_0)$$