

# Complex Analysis - Part 6

(1)  $f: \mathbb{C} \rightarrow \mathbb{C}$  is (complex) differentiable at  $z_0 \in \mathbb{C}$  if

there is  $f'(z_0) \in \mathbb{C}$  and a function  $\psi: \mathbb{C} \rightarrow \mathbb{C}$  with:

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + \psi(z) \quad \text{where} \quad \frac{\psi(z)}{z - z_0} \xrightarrow{z \rightarrow z_0} 0$$

(2)  $f_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is called (totally) differentiable at  $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$  if

there is a matrix  $J \in \mathbb{R}^{2 \times 2}$  and a map  $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with:

$$f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f_{\mathbb{R}}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + J \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + \phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \quad \text{where} \quad \frac{\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)}{\left\|\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right\|} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} 0$$

Question: In which cases does a matrix-vector multiplication represent a multiplication of complex numbers?

Let's check: 
$$\underbrace{(a+ib)}_W \cdot \underbrace{(x+iy)}_Z = (a \cdot x - b \cdot y) + i \cdot (b \cdot x + a \cdot y)$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \cdot x - b \cdot y \\ b \cdot x + a \cdot y \end{pmatrix}$$

Theorem:  $f: \mathbb{C} \rightarrow \mathbb{C}$  is (complex) differentiable at  $z_0 = x_0 + iy_0 \in \mathbb{C}$

$\Leftrightarrow f_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is (totally) differentiable at  $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$

and the Jacobian matrix at  $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$  has the form:  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$\Leftrightarrow$  For  $f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$  the Cauchy-Riemann equations are satisfied:

two maps:  $\mathbb{R}^2 \rightarrow \mathbb{R}$   $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$  at point  $(x_0, y_0)$