Complex Analysis – Part 35

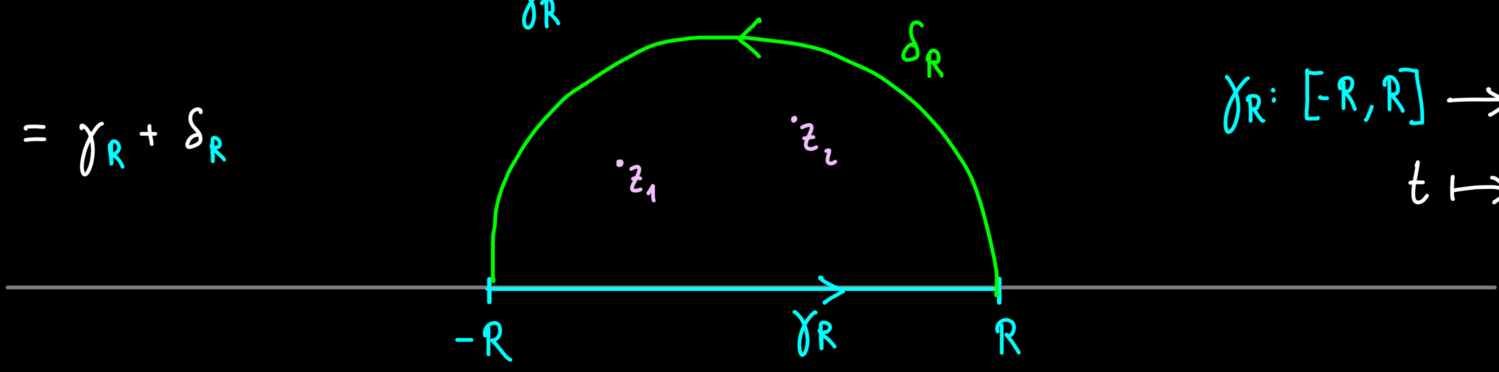
$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\, dx = \,?$$

$$= \lim_{R\to\infty} \underbrace{\int_{-R}^{-R} f(x)\, dx}_{=} \qquad \text{where} \quad f(x) = \frac{x^4}{1+x^6}$$

complex contour integral:

$$\int_{\gamma_R} f(z)\, dz \qquad \text{where} \quad f(z) = \frac{z^4}{1+z^6}$$

$$\Gamma_R = \gamma_R + \delta_R$$



$$\gamma_R : [-R, R] \to \mathbb{C}$$
$$t \mapsto t$$

residue theorem:

$$\oint_{\Gamma_R} f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)$$

$$= \int_{\gamma_R} f(z)\, dz + \int_{\delta_R} f(z)\, dz \qquad \text{where} \quad \left| \int_{\delta_R} f(z)\, dz \right| \leq \max_{z\in \operatorname{Ran}(\delta_R)} \underbrace{|f(z)|}_{= \frac{(Re^{it})^4}{1+(Re^{it})^6}} \cdot \operatorname{length}(\delta_R)$$

$$\leq C \cdot \frac{1}{R^{\cancel{2}}} \cdot \pi \cdot \cancel{R} \xrightarrow{R\to\infty} 0$$

Hence:

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty} \oint_{\Gamma_R} f(z)\,dz$$

$$= 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{Res}(f,z)$$

poles: $1+x^6 = 0 \implies z_1 = e^{i\frac{\pi}{6}}, \quad z_2 = e^{3i\frac{\pi}{6}}, \quad z_3 = e^{5i\frac{\pi}{6}}$

formula for simple poles: $\mathrm{Res}\left(\frac{h}{g}, z\right) = \frac{h(z)}{g'(z)}$

$$\implies \int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx = 2\pi i \cdot \left(\frac{1}{6} e^{-i\frac{\pi}{6}} + \frac{1}{6} e^{-3i\frac{\pi}{6}} + \frac{1}{6} e^{-5i\frac{\pi}{6}} \right)$$

$$= \frac{1}{3}\pi i \left(i \underbrace{\sin\left(-\frac{\pi}{6}\right)}_{-\sin\left(\frac{\pi}{6}\right)} + i \underbrace{\sin\left(-\frac{3\pi}{6}\right)}_{=-1} + i \underbrace{\sin\left(-\frac{5\pi}{6}\right)}_{-\sin\left(\frac{\pi}{6}\right)} \right)$$

$$= \underline{\frac{1}{3}\pi \left(1 + 2\sin\left(\frac{\pi}{6}\right)\right)}$$