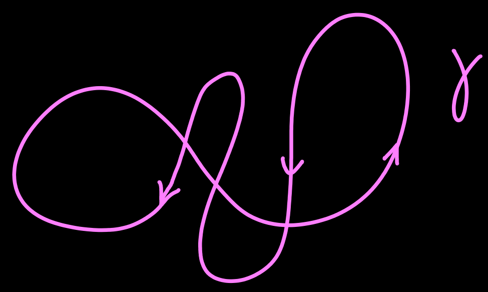
# Complex Analysis – Part 32

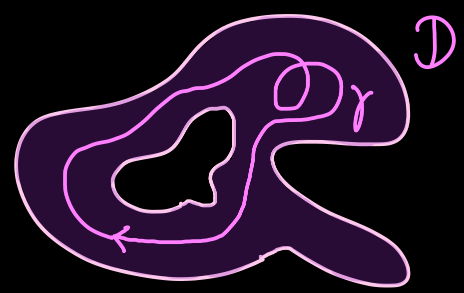
Residue $\rightsquigarrow$ Residue Theorem




Short recapitulation: Closed curve integrals:

$f : \mathcal{D} \longrightarrow \mathbb{C}$ holomorphic.

(1) $F : \mathcal{D} \longrightarrow \mathbb{C}$ antiderivative of $f$ $\quad (F' = f)$



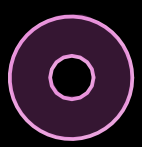

$$\Longrightarrow \oint_\gamma f(z)\, dz = 0$$

(2) $\mathcal{D}$ star domain or $\mathcal{D} =$ 

$$\Longrightarrow \oint_\gamma f(z)\, dz = 0$$

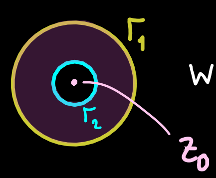
(3) $\mathcal{D} = \mathbb{C} \setminus \{z_0\}$ , $f(z) = \frac{1}{z - z_0}$ $\Longrightarrow \oint_\gamma f(z)\, dz = 2\pi i \cdot \text{wind}(\gamma, z_0)$

Combine (1) and (3) for Laurent series:

$\mathcal{D} =$  $= \{ z \in \mathbb{C} \mid r_2 < |z - z_0| < r_1 \}$ , $f(z) = \sum_{k=-\infty}^{\infty} a_k \cdot (z - z_0)^k$

$$\Longrightarrow \oint_\gamma f(z)\, dz = a_{-1} \oint_\gamma (z - z_0)^{-1}\, dz = \underbrace{a_{-1}}_{\text{Res}(f, z_0)} \cdot 2\pi i \cdot \text{wind}(\gamma, z_0)$$

residue

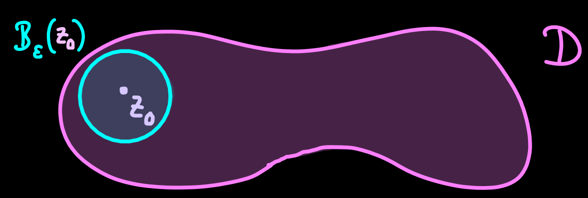
Fact: Let $f$ be a Laurent series defined on  with $r_2 < r < r_1$ .



Then: $\text{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} f(z)\, dz$

Definition: Let $f : \mathcal{D} \longrightarrow \mathbb{C}$ be holomorphic and $z_0$ be an isolated singularity of $f$.

If $\overline{B_\varepsilon(z_0)} \setminus \{z_0\} \subseteq \mathcal{D}$ , then we define:




$$\text{Res}(f, z_0) := \frac{1}{2\pi i} \oint_{\partial B_\varepsilon(z_0)} f(z)\, dz$$

residue of $f$ at $z_0$