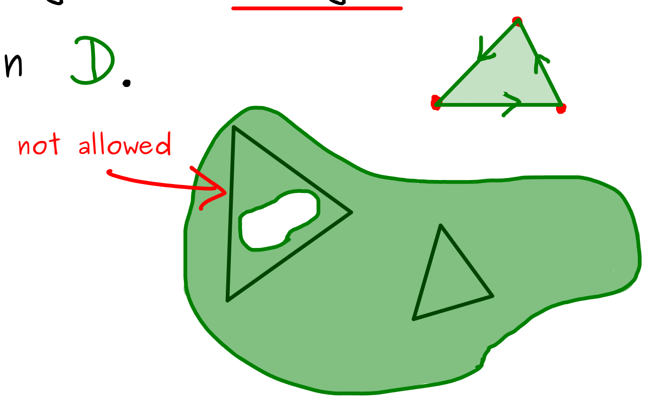
Complex Analysis – Part 22

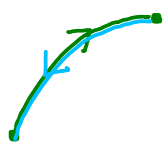
Goursat's theorem: $f: D \to \mathbb{C}$ holomorphic,

$\gamma: [a,b] \longrightarrow D$ closed curve where the image is a triangle and the inner part lies in D.

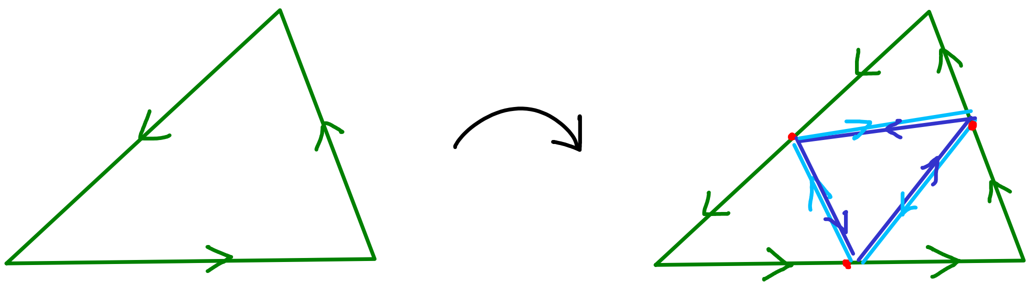


Then: $$\oint_\gamma f(z)\, dz = 0$$

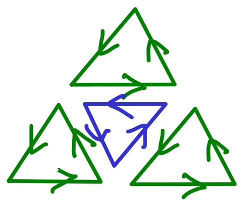
Proof:

Basic idea:  $$0 = \int_{\nearrow} + \int_{\swarrow} = \oint_{\nearrow + \swarrow}$$

Decompose triangle:



$$\oint_\gamma f(z)\, dz = \oint_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)\, dz$$



$$= \oint_{\gamma_1} f(z)\, dz + \oint_{\gamma_2} f(z)\, dz + \oint_{\gamma_3} f(z)\, dz + \oint_{\gamma_4} f(z)\, dz$$

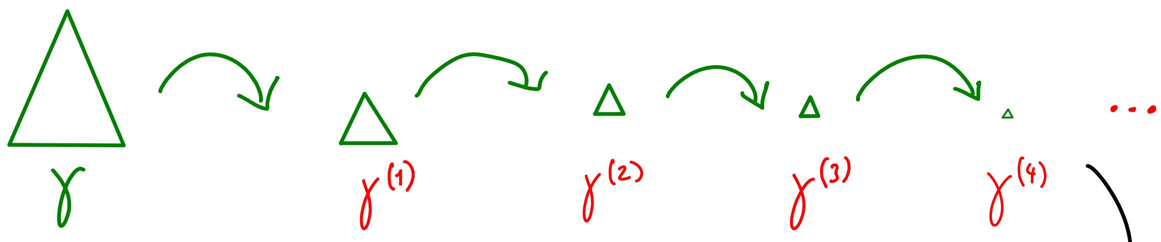
$$\left| \oint_\gamma f(z)\, dz \right| \leq \left| \oint_{\gamma_j} f(z)\, dz \right| + \left| \oint_{\gamma_j} f(z)\, dz \right| + \left| \oint_{\gamma_j} f(z)\, dz \right| + \left| \oint_{\gamma_j} f(z)\, dz \right|$$

$$= 4 \cdot \left| \oint_{\gamma^{(1)}} f(z)\, dz \right|$$

γ_j represents maximal value

$= \gamma^{(1)}$

Repeat n times:



converge to a point $n \to \infty$

z_0 lies in the interior of all $\gamma^{(n)}$

$$\left| \oint_\gamma f(z)\, dz \right| \leq 4^n \cdot \left| \oint_{\gamma^{(n)}} f(z)\, dz \right| \qquad (*)$$

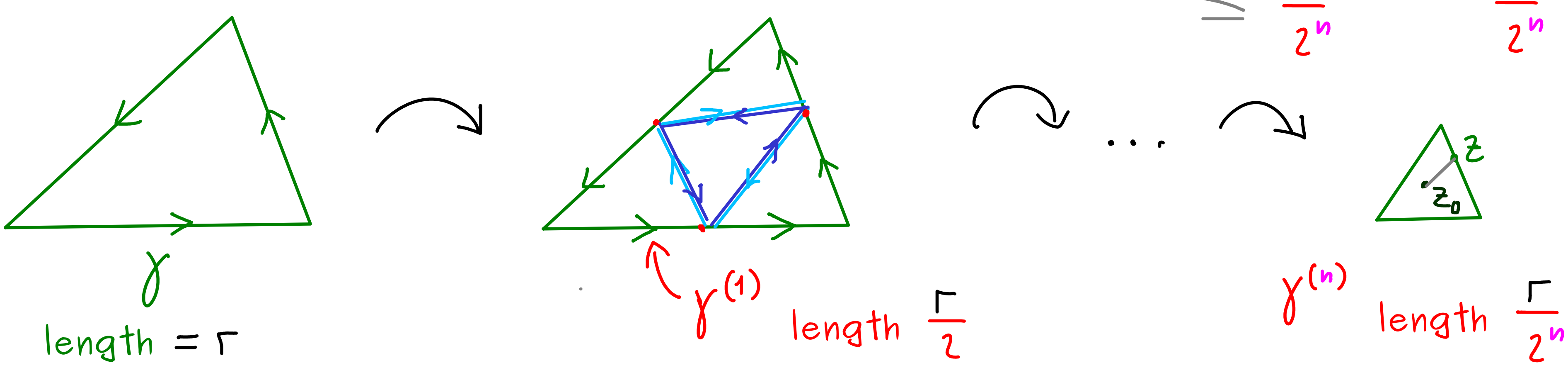
Complex differentiability at z_0:

$$f(z) = \underbrace{f(z_0) + f'(z_0)\cdot(z - z_0)}_{\text{has antiderivative } \Rightarrow \oint = 0} + \varphi(z) \qquad \text{where } \frac{\varphi(z)}{z - z_0} \xrightarrow{z \to z_0} 0$$

$$\varphi(z) = \psi(z)(z - z_0) \quad \text{with } \psi(z) \xrightarrow{z \to z_0} 0$$

$$\left| \oint_{\gamma^{(n)}} f(z)\, dz \right| = \left| \oint_{\gamma^{(n)}} \varphi(z)\, dz \right| \leq \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\varphi(z)| \cdot \text{length}(\gamma^{(n)})$$

$$\leq \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\psi(z)| \cdot \underbrace{\max_{z \in \mathrm{Ran}(\gamma^{(n)})} |z - z_0|}_{\leq \frac{r}{2^n}} \cdot \underbrace{\text{length}(\gamma^{(n)})}_{\frac{r}{2^n}}$$



$$\left| \oint_\gamma f(z)\, dz \right| \overset{(*)}{\leq} 4^n \cdot \left| \oint_{\gamma^{(n)}} f(z)\, dz \right| \leq r^2 \cdot \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\psi(z)| \xrightarrow{n \to \infty} 0$$

$\square$