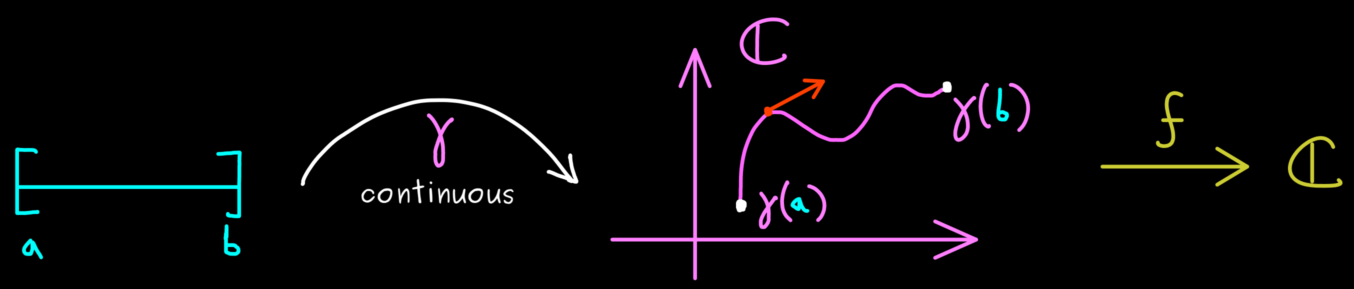
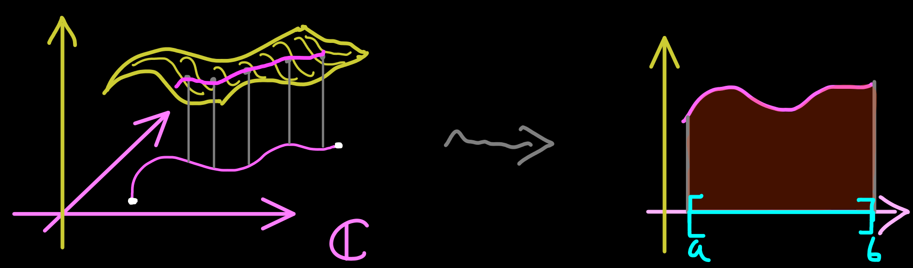
Complex Analysis – Part 18



First look at $f: \mathbb{C} \to \mathbb{R}$

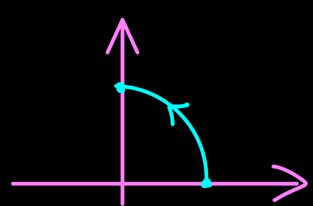


Definition: For a parametrized curve $\gamma: [a,b] \to \mathbb{C}$ continuously differentiable with $\gamma': [a,b] \to \mathbb{C}$, we define:

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

for continuous functions $f: U \to \mathbb{C}$ with $\mathrm{Ran}(\gamma) \subseteq U$.

Examples: (a) $f(z) = z$, $\gamma_1: [0, \frac{\pi}{2}] \to \mathbb{C}$
$t \mapsto e^{it}$

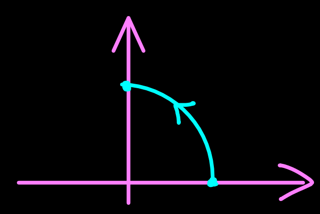


$$\int_{\gamma_1} f(z)\, dz = \int_0^{\frac{\pi}{2}} f(\underbrace{\gamma_1(t)}_{e^{it}}) \cdot \underbrace{\gamma_1'(t)}_{i \cdot e^{it}}\, dt = i \cdot \int_0^{\frac{\pi}{2}} e^{2it}\, dt = i \cdot \frac{1}{2i} e^{2it} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \cdot \left(e^{i\pi} - 1\right) = -1$$

(b) $f(z) = z$, $\gamma_2 : [0,1] \longrightarrow \mathbb{C}$

$t \longmapsto e^{i\frac{\pi}{2}t}$

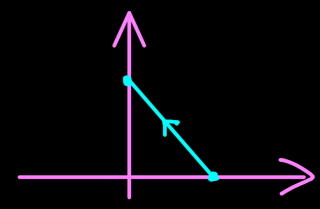


$$\int_{\gamma_2} f(z)\,dz = \int_0^1 f(\underbrace{\gamma_2(t)}_{e^{i\frac{\pi}{2}t}}) \cdot \underbrace{\gamma_2'(t)}_{i\frac{\pi}{2}e^{i\frac{\pi}{2}t}}\,dt = i \cdot \frac{\pi}{2} \int_0^1 e^{i\pi t}\,dt = i\frac{\pi}{2}\,\frac{1}{i\pi}\,e^{i\pi t}\Big|_0^1$$

$$= -1$$

(c) $f(z) = z$, $\gamma_3 : [0,1] \longrightarrow \mathbb{C}$

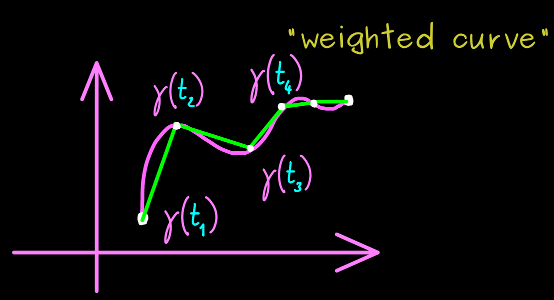
$t \longmapsto (1-t) + i \cdot t$



$$\int_{\gamma_3} f(z)\,dz = \int_0^1 f(\gamma_3(t)) \cdot \underbrace{\gamma_3'(t)}_{(-1+i)}\,dt = (-1+i)\int_0^1 \big(1 + (i-1)\,t\big)\,dt$$

$$= (-1+i)\left(t + \tfrac{1}{2}(i-1)\,t^2\right)\Big|_0^1 = (-1+i)\left(1 + \tfrac{1}{2}(i-1)\right) = -1$$

Another visualisation:



$$\sum_{i=1}^{n} f(\gamma(t_i)) \cdot \big(\gamma(t_{i+1}) - \gamma(t_i)\big)$$

$$= \sum_{i=1}^{n} f(\gamma(t_i))\,\frac{\gamma(t_{i+1}) - \gamma(t_i)}{t_{i+1} - t_i}\,(t_{i+1} - t_i)$$

$$\xrightarrow[\text{(in some sense)}]{n\to\infty} \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$