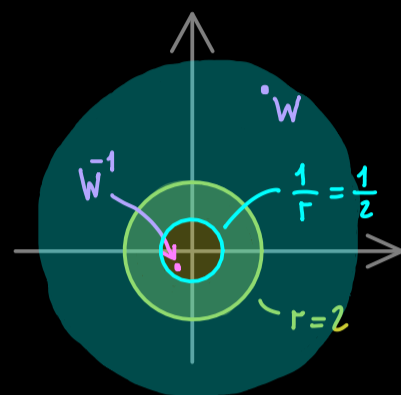# Complex Analysis — Part 15

Laurent series   (generalisation of power series + holomorphic)

$$\sum_{k=0}^{\infty} a_k \cdot z^k \quad \text{with radius of convergence} \quad r \in [0, \infty]$$

$$\sum_{k=0}^{\infty} a_k \cdot \left(\frac{1}{w}\right)^k \quad \text{is convergent} \quad \begin{cases} \left|\frac{1}{w}\right| < r \\ \iff \\ |w| > \frac{1}{r} \end{cases}$$
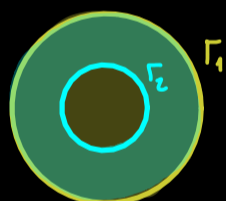

$\overset{\text{chain rule}}{\Longrightarrow} \quad w \mapsto \sum_{k=0}^{\infty} a_k \cdot w^{-k} \quad \text{is holomorphic on} \quad \mathbb{C} \setminus \overline{\mathcal{B}_{\frac{1}{r}}(0)}$

$$\left(\text{alternatively:} \quad \text{constant} + \sum_{k=-1}^{-\infty} b_k \cdot z^k \right)$$

Combine two series:

$z \mapsto \sum_{k=0}^{\infty} a_k \cdot z^k \rightsquigarrow \quad \text{with radius of convergence} \quad r_1$

$z \mapsto \sum_{k=-1}^{-\infty} b_k \cdot z^k \quad \overset{\sum_{k=1}^{\infty} b_{-k} \cdot z^k \rightsquigarrow \text{ with radius of convergence } r}{\rightsquigarrow \quad \text{with "radius of convergence"} \quad r_2 = \frac{1}{r}}$

Definition: A <u>Laurent series</u> written as $\quad \sum_{k=-\infty}^{\infty} a_k \cdot (z - z_0)^k \quad$ is a pair of two series:

$z \mapsto \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k \quad \text{with radius of convergence} \quad r_1 \in [0, \infty]$

principal part
$\rightsquigarrow z \mapsto \sum_{k=-1}^{-\infty} a_k \cdot (z - z_0)^k \quad \text{with "radius of convergence"} \quad r_2 \in [0, \infty]$

a_{-1} is called the <u>residue</u> of the Laurent series.

The Laurent series is a holomorphic function on $\left\{ z \in \mathbb{C} \mid r_2 < |z - z_0| < r_1 \right\}$