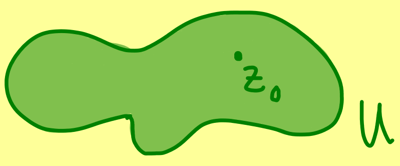
The Bright Side of Mathematics

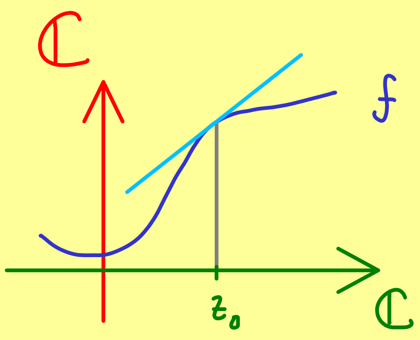
Complex Analysis – Part 3

$U \subseteq \mathbb{C}$ open, $z_0 \in U$.



$f: U \to \mathbb{C}$ is (complex) differentiable at z_0

$$:\Longleftrightarrow \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$



$\Longleftrightarrow$ there is a function: $\Delta_{f,z_0}: U \longrightarrow \mathbb{C}$ with

$$f(z) = f(z_0) + (z - z_0) \cdot \Delta_{f,z_0}(z) \quad \text{for all } z \in U$$

and Δ_{f,z_0} is continuous at z_0.

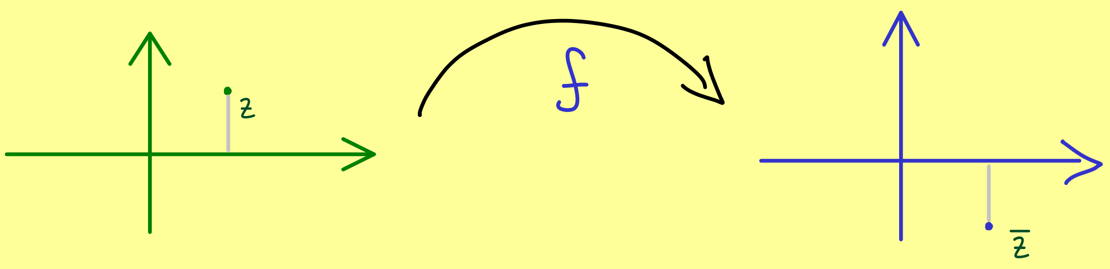
Definition: $f'(z_0) := \Delta_{f,z_0}(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ is called the (complex) derivative of f at z_0.

Examples: (a) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = m \cdot z + c$ for $m, c \in \mathbb{C}$

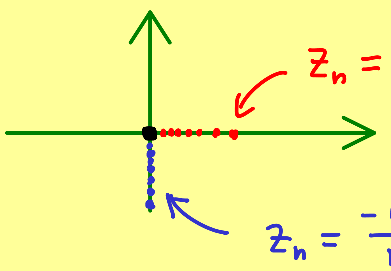
$$f(z) = \underbrace{(m \cdot z_0 + c)}_{f(z_0)} + (z - z_0) \cdot \underbrace{m}_{\Delta_{f,z_0}(z)} \Rightarrow f'(z_0) = m$$

(b) $f: \mathbb{C} \to \mathbb{C}$

$z \longmapsto \bar{z}$



differentiable at $z_0 = 0$? $\quad f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \dfrac{\bar{z}}{z}$ does not exist!



$z_n = \frac{1}{n}$: $\quad \dfrac{\bar{z}_n}{z_n} = \dfrac{\frac{1}{n}}{\frac{1}{n}} = 1 \xrightarrow{n \to \infty} 1$

$\neq$

$z_n = \frac{-i}{n}$: $\quad \dfrac{\bar{z}_n}{z_n} = \dfrac{\frac{i}{n}}{\frac{-i}{n}} = -1 \xrightarrow{n \to \infty} -1 \quad \Rightarrow f$ is not differentiable at 0