The Bright Side of Mathematics

The following pages cover the whole Complex Analysis course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

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The Bright Side of Mathematics

Complex Analysis – Part 1

analysis of differentiable functions $f: \mathbb{C} \to \mathbb{C}$

(instead of $f: \mathbb{R} \to \mathbb{R}$)

$\mathbb{R} \subseteq \mathbb{C}$ $\Rightarrow$ helpful for real problems like $\displaystyle\int_{-\infty}^{\infty} \frac{x \cdot \sin(x)}{1+x^2}\,dx = \frac{\pi}{e}$

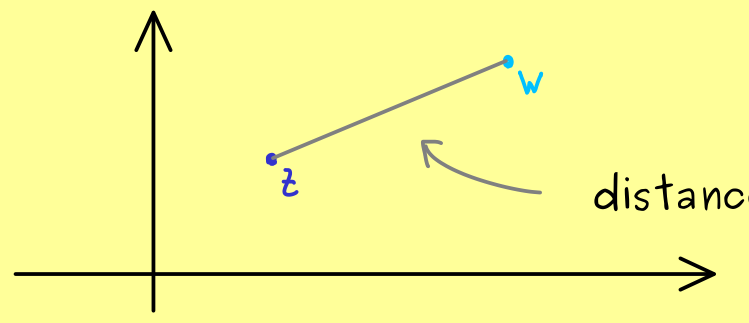
We need:
- sets
- complex numbers

} Start Learning Mathematics

- basic knowledge of continuous and differentiable functions
- basic knowledge of power series

Real Analysis (some videos)

Some definitions: $\mathbb{C}$ is a set with a distance (metric space)



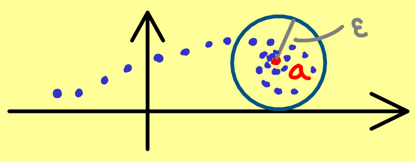
distance: $|z-w|$

absolute value in $\mathbb{C}$

A sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is convergent to $a \in \mathbb{C}$

$$\Longleftrightarrow \quad (|z_n - a|)_{n\in\mathbb{N}} \subseteq \mathbb{R} \text{ is convergent to } 0$$

$$\Longleftrightarrow \quad \forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N : \;\; |z_n - a| < \varepsilon$$



ε-ball: $B_\varepsilon(a) := \{ w \in \mathbb{C} \mid |w-a| < \varepsilon \}$

A function $f: \mathbb{C} \to \mathbb{C}$ is continuous at $z_0 \in \mathbb{C}$ if for all sequences $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$:

$z_n \xrightarrow{n\to\infty} z_0$ implies $f(z_n) \xrightarrow{n\to\infty} f(z_0)$.

means: $(z_n)_{n\in\mathbb{N}}$ is convergent to z_0

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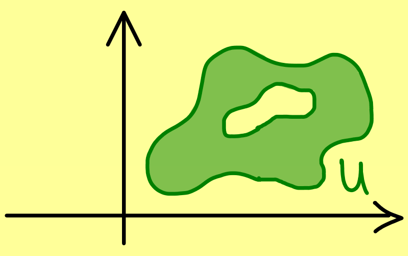
Complex Analysis – Part 2

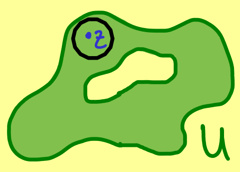
$f: \mathbb{C} \to \mathbb{C}$ differentiable at z_0?

domain can be any open set $U \subseteq \mathbb{C}$



Definition: $U \subseteq \mathbb{C}$ is called open if

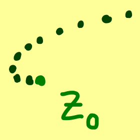
$$\forall z \in U \;\; \exists \varepsilon > 0 \; : \; B_\varepsilon(z) \subseteq U$$



Definition: $U \subseteq \mathbb{C}$ open, $z_0 \in U$. $f: U \to \mathbb{C}$ is called

(complex) differentiable at $z_0 \in U$ if

$$\underbrace{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$ exists.

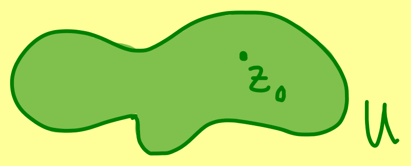


For all sequences $(z_n)_{n \in \mathbb{N}} \subseteq U \setminus \{z_0\}$ with $z_n \xrightarrow{n \to \infty} z_0$,

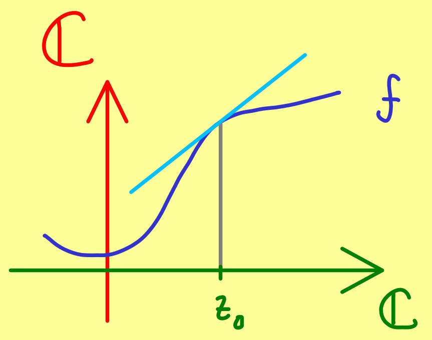
the sequence $\dfrac{f(z_n) - f(z_0)}{z_n - z_0}$ converges (to the same number).

Complex Analysis – Part 3

$U \subseteq \mathbb{C}$ open, $z_0 \in U$.



$f: U \to \mathbb{C}$ is (complex) differentiable at z_0



$:\Longleftrightarrow \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists.

$\Longleftrightarrow$ there is a function: $\Delta_{f,z_0}: U \longrightarrow \mathbb{C}$ with

$f(z) = f(z_0) + (z - z_0) \cdot \Delta_{f,z_0}(z)$ for all $z \in U$

and Δ_{f,z_0} is continuous at z_0.

Definition: $f'(z_0) := \Delta_{f,z_0}(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ is called

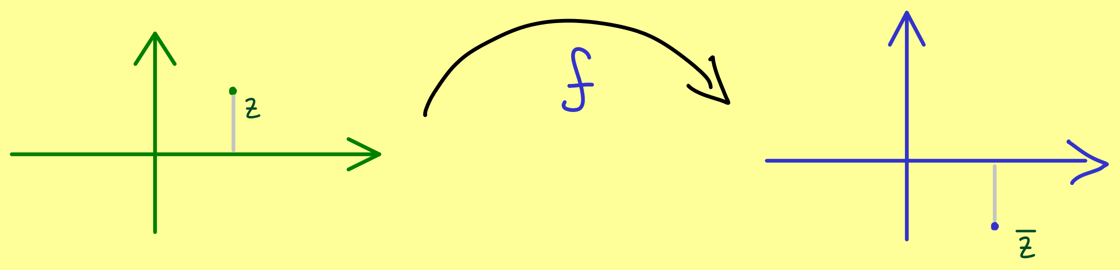
the (complex) derivative of f at z_0.

Examples: (a) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = m \cdot z + c$ for $m, c \in \mathbb{C}$

$f(z) = \underbrace{(m \cdot z_0 + c)}_{f(z_0)} + (z - z_0) \cdot \underbrace{m}_{\Delta_{f,z_0}(z)} \Rightarrow f'(z_0) = m$

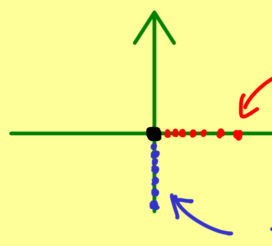
(b) $f: \mathbb{C} \to \mathbb{C}$

$z \longmapsto \bar{z}$



differentiable at $z_0 = 0$? $f'(0) = \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z - 0} = \lim\limits_{z \to 0} \dfrac{\bar{z}}{z}$

does not exist!



$z_n = \frac{1}{n}$: $\dfrac{\bar{z}_n}{z_n} = \dfrac{\frac{1}{n}}{\frac{1}{n}} = 1 \xrightarrow{n \to \infty} 1$

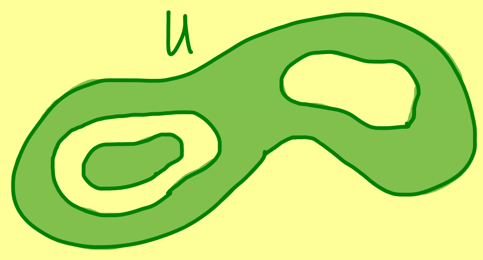
$\neq$

$z_n = \frac{-i}{n}$: $\dfrac{\bar{z}_n}{z_n} = \dfrac{\frac{i}{n}}{\frac{-i}{n}} = -1 \xrightarrow{n \to \infty} -1$ $\Rightarrow f$ is not differentiable at 0

Complex Analysis – Part 4

(regular/ (complex) analytic/...)

Definition: $U \subseteq \mathbb{C}$ open. $f: U \to \mathbb{C}$ is called holomorphic (on U)



if f is (complex) differentiable at every $z_0 \in U$.

If $U = \mathbb{C}$, the holomorphic function is called entire.

Properties: (a) f is holomorphic $\Rightarrow$ f is continuous

(b) $f, g: U \to \mathbb{C}$ holomorphic $\Rightarrow$ $f+g$, $f \cdot g$ holomorphic

(c) Sum rule, product rule, quotient rule and chain rule for derivatives hold.

Examples: (1) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = a_m \cdot z^m + a_{m-1} z^{m-1} + \cdots + a_1 \cdot z^1 + a_0$

A polynomial is an entire function. with $a_0, \dots, a_m \in \mathbb{C}$

$$f'(z) = m \cdot a_m \cdot z^{m-1} + (m-1) \cdot a_{m-1} z^{m-2} + \cdots + 2 \cdot a_2 \cdot z^1 + a_1$$

(2) $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = \frac{1}{z}$ is holomorphic

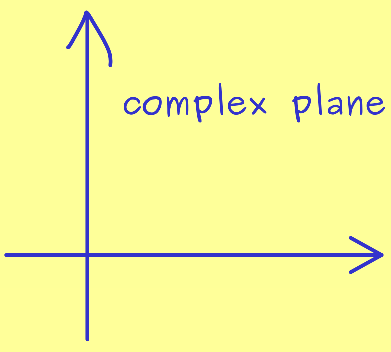
(3) $f: \mathbb{C} \setminus \underbrace{S}_{= \{z \in \mathbb{C} \mid q(z) = 0\}} \to \mathbb{C}$, $f(z) = \dfrac{p(z)}{q(z)}$ is holomorphic

polynomial

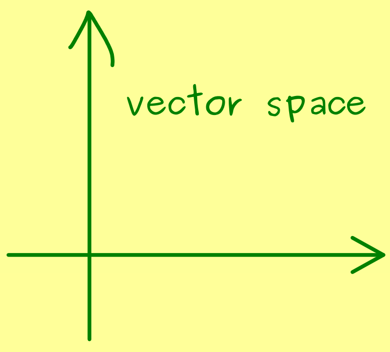
polynomial

Complex Analysis – Part 5

$\mathbb{C}$:



$\mathbb{R}^2$:



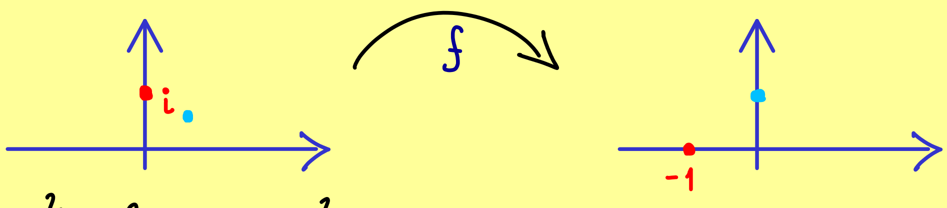
$\mathbb{C}$ is $\mathbb{R}^2$ with a multiplication

Remember: Each map $f: \mathbb{C} \to \mathbb{C}$ induces a map $f_\mathbb{R}: \mathbb{R}^2 \to \mathbb{R}^2$ (and vice versa)

Example: $f: \mathbb{C} \to \mathbb{C}$

$$z \mapsto z^2$$

$$x + iy \mapsto (x+iy)^2 = x^2 + 2ixy - y^2$$



$$f_\mathbb{R}: \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \qquad f_\mathbb{R}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Definition: A map $f_\mathbb{R}: \mathbb{R}^2 \to \mathbb{R}^2$ is called (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ if there is a matrix $J \in \mathbb{R}^{2\times 2}$ and a map $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ with:

$$f_\mathbb{R}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \underbrace{f_\mathbb{R}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + J\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)}_{\text{linear approximation}} + \phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where
$$\frac{\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} 0$$

$\sqrt{(x-x_0)^2 + (y-y_0)^2} =$ (Euclidean) norm

J is called the Jacobian matrix of $f_\mathbb{R}$ at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$.

$$J = \begin{pmatrix} | & | \\ \frac{\partial f_\mathbb{R}}{\partial x} & \frac{\partial f_\mathbb{R}}{\partial y} \\ | & | \end{pmatrix} \quad \text{(evaluate at } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\text{)}$$

Example: $f_\mathbb{R}: \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$

$$J = \begin{pmatrix} 2\cdot x & -2\cdot y \\ 2\cdot y & 2\cdot x \end{pmatrix}$$

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Complex Analysis – Part 6

(1) $f: \mathbb{C} \to \mathbb{C}$ is (complex) differentiable at $z_0 \in \mathbb{C}$ if there is $f'(z_0) \in \mathbb{C}$ and a function $\varphi: \mathbb{C} \to \mathbb{C}$ with:

$$f(z) = f(z_0) + f'(z_0) \cdot (z - z_0) + \varphi(z) \quad \text{where} \quad \frac{\varphi(z)}{z - z_0} \xrightarrow{z \to z_0} 0$$

(2) $f_{\mathbb{R}}: \mathbb{R}^2 \to \mathbb{R}^2$ is called (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ if there is a matrix $J \in \mathbb{R}^{2\times 2}$ and a map $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ with:

$$f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f_{\mathbb{R}}\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + J\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + \phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \quad \text{where} \quad \frac{\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} 0$$

Question: In which cases does a matrix-vector multiplication represent a multiplication of complex numbers?

Let's check: $\underset{(a+ib)}{\overset{\shortparallel}{w}} \cdot \underset{(x+iy)}{\overset{\shortparallel}{z}} = (a \cdot x - by) + i \cdot (bx + ay)$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \cdot x - by \\ bx + ay \end{pmatrix}$$

Theorem: $f: \mathbb{C} \to \mathbb{C}$ is (complex) differentiable at $z_0 = x_0 + i y_0 \in \mathbb{C}$

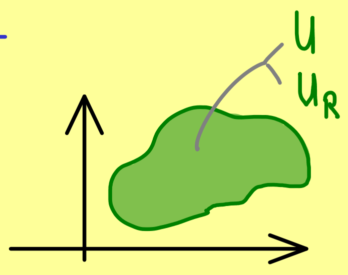
$\Longleftrightarrow$ $f_{\mathbb{R}}: \mathbb{R}^2 \to \mathbb{R}^2$ is (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ and the Jacobian matrix at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ has the form: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$\Longleftrightarrow$ For $f_{\mathbb{R}}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ the Cauchy-Riemann equations are satisfied:

two maps: $\mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at point } (x_0, y_0)$$

Complex Analysis – Part 7



Theorem: $U \subseteq \mathbb{C}$ open.

$f: U \to \mathbb{C}$ is holomorphic

$\Longleftrightarrow$ Real part of f as a function on $U_\mathbb{R} \subseteq \mathbb{R}^2$

$u: U_\mathbb{R} \to \mathbb{R}$

and imaginary part of f as a function on $U_\mathbb{R} \subseteq \mathbb{R}^2$

$v: U_\mathbb{R} \to \mathbb{R}$

fulfil: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at all points $(x,y) \in U_\mathbb{R}$

Examples: (a) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z$ $\Rightarrow$ $f(x+iy) = \underbrace{x}_{u(x,y)} + i\underbrace{y}_{v(x,y)}$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 0$$
$$\frac{\partial v}{\partial y} = 1 \qquad -\frac{\partial v}{\partial x} = 0 \quad \Rightarrow f \text{ is holomorphic}$$

(b) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$ $\Rightarrow$ $f(x+iy) = \underbrace{x}_{u(x,y)} + i\underbrace{(-y)}_{v(x,y)}$

$$\frac{\partial u}{\partial x} = 1 \;\neq\; \frac{\partial v}{\partial y} = -1 \quad \Rightarrow f \text{ is not holomorphic}$$

(c) $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 + iz$ $\Rightarrow$ $f(x+iy) = (x+iy)^2 + i(x+iy)$

$$= x^2 + i2xy - y^2 + ix - y$$
$$= \underbrace{(x^2 - y^2 - y)}_{u(x,y)} + i\underbrace{(2xy + x)}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y - 1$$
$$\frac{\partial v}{\partial y} = 2x \qquad -\frac{\partial v}{\partial x} = -(2y+1)$$

$\Rightarrow f$ is holomorphic

Complex Analysis – Part 8

$f: U \to \mathbb{C}$ holomorphic

$\frac{\partial f}{\partial z}(z_0)$ ← Wirtinger derivatives → $\frac{\partial f}{\partial \bar z}(z_0)$

$\frac{\partial f}{\partial z}(z_0) \overset{?}{=} f'(z_0)$, $\quad \frac{\partial f}{\partial \bar z}(z_0) \overset{?}{=} 0$

$$f'(x+iy) = \underbrace{a}_{\frac{\partial u}{\partial x}(x,y)} + i\underbrace{b}_{\frac{\partial v}{\partial x}(x,y)}$$

for $f_\mathbb{R}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$

and map $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto f(x+iy) = u(x,y) + i\, v(x,y)$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial v}{\partial y}} + i\underbrace{\frac{\partial v}{\partial x}}_{-\frac{\partial u}{\partial y}}\right)$$

$$= \frac{1}{2}\left(\underbrace{\frac{\partial}{\partial x}(u+iv)}_{\frac{\partial f}{\partial x}} - i\underbrace{\frac{\partial}{\partial y}(u+iv)}_{\frac{\partial f}{\partial y}}\right)$$

Definition: $\frac{\partial}{\partial z} := \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\quad \frac{\partial}{\partial \bar z} := \frac{1}{2}\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Example: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\cdot 2\cdot x\cdot y \Rightarrow \frac{\partial f}{\partial x} = 2\cdot x + i 2 y = 2\cdot z$

$\frac{\partial f}{\partial y} = -2y + i2x = 2\cdot i\cdot z$

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(2z + i\cdot 2iz\right) = 0 \ , \quad \frac{\partial f}{\partial z} = \frac{1}{2}\left(2z - i\cdot 2iz\right) = 2\cdot z$$

Fact: $f: U \to \mathbb{C}$ holomorphic $\iff$ $\frac{\partial f}{\partial \bar z} = 0$ at all points in U

In this case: $f' = \frac{\partial f}{\partial z}$

Complex Analysis – Part 9

Power series

Example: Exponential function: $\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$

Definition: For a sequence of complex numbers $a_0, a_1, a_2, a_3, \dots$,

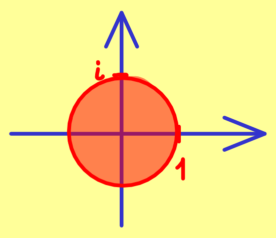
the function $f: D \to \mathbb{C}$, $z \mapsto \sum_{k=0}^{\infty} a_k (z - z_0)^k$ expansion point

with $D := \left\{ z \in \mathbb{C} \;\middle|\; \sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ is convergent} \right\}$

is called a power series.

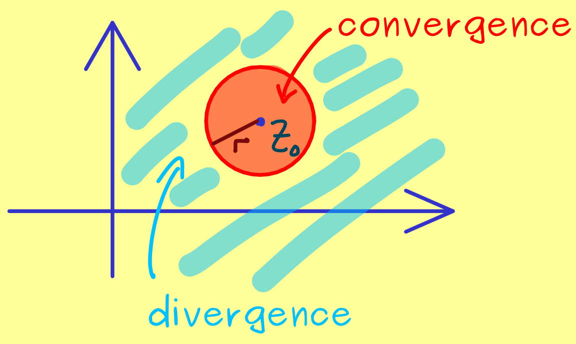
Example: Geometric series: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ for $|z| < 1$

$D = B_1(0)$ divergent for $|z| \geq 1$



Fact: For $\sum_{k=0}^{\infty} a_k (z - z_0)^k$, there is a maximal $r \in [0, \infty) \cup \{\infty\}$

such that $\begin{cases} B_r(z_0) \subseteq D & \text{for } r \in [0, \infty) \\ \mathbb{C} = D & \text{for } r = \infty \end{cases}$



and for $z \in \mathbb{C} \setminus \overline{B_r(z_0)}$ the power series is divergent.

Cauchy-Hadamard: $\frac{1}{r} = \limsup_{k \to \infty} \sqrt[k]{|a_k|} \in [0, \infty) \cup \{\infty\}$ $\left(\frac{1}{0} = \infty, \; \frac{1}{\infty} = 0 \right)$

r is called the radius of convergence.

Complex Analysis – Part 10

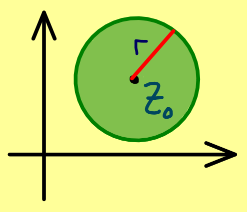
Definition: A sequence of functions $f_n : U \to \mathbb{C}$ $(n \in \mathbb{N})$

is uniformly convergent to $f : U \to \mathbb{C}$

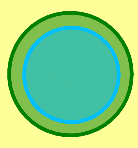
if $\underbrace{\| f_n - f \|_\infty}_{:= \sup_{z \in U} |f_n(z) - f(z)|} \xrightarrow{n \to \infty} 0$.

Result for power series: Let $f : B_r(z_0) \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k$

be a power series with radius of convergence $r > 0$.



Then: (1) $\sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k$ is uniformly convergent on $\overline{B_c(z_0)}$ with $c < r$



(sequence of functions $f_n : \overline{B_c(z_0)} \to \mathbb{C}$, $f_n(z) = \sum_{k=0}^{n} a_k \cdot (z - z_0)^k$ is uniformly convergent)

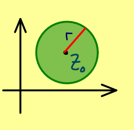
(2) $\sum_{k=1}^{\infty} a_k \cdot k (z - z_0)^{k-1}$ is uniformly convergent on $\overline{B_c(z_0)}$ with $c < r$

(sequence of functions $f_n' : \overline{B_c(z_0)} \to \mathbb{C}$, $f_n'(z) = \sum_{k=0}^{n} a_k \cdot k (z - z_0)^{k-1}$ is uniformly convergent)

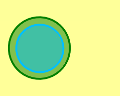
(3) f is complex differentiable with $f'(z) = \sum_{k=1}^{\infty} a_k \cdot k (z - z_0)^{k-1}$

Complex Analysis – Part 11

Result for power series: Let $f: B_r(z_0) \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} a_k \cdot (z-z_0)^k$ be a power series with radius of convergence $r > 0$.



Then: (1) $\sum_{k=0}^{\infty} a_k \cdot (z-z_0)^k$ is uniformly convergent on $\overline{B_c(z_0)}$ with $c < r$



(2) $\sum_{k=1}^{\infty} a_k \cdot k (z-z_0)^{k-1}$ is uniformly convergent on $\overline{B_c(z_0)}$ with $c < r$

(3) f is complex differentiable with $f'(z) = \sum_{k=1}^{\infty} a_k \cdot k (z-z_0)^{k-1}$

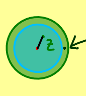
Proof: Assume $z_0 = 0$. $f_n: \overline{B_c(0)} \to \mathbb{C}$, $f_n(z) = \sum_{k=0}^{n} a_k \cdot z^k$

(1) $$\|f - f_n\|_\infty = \sup_{z \in \overline{B_c(0)}} \left| \sum_{k=n+1}^{\infty} a_k \cdot z^k \right| = \sup_{z \in \overline{B_c(0)}} \lim_{N \to \infty} \left| \sum_{k=n+1}^{N} a_k \cdot z^k \right|$$

supremum norm on $\overline{B_c(0)}$

Δ-inequality

$$\leq \sup_{z \in \overline{B_c(0)}} \lim_{N \to \infty} \sum_{k=n+1}^{N} |a_k| \cdot \underbrace{|z|}_{\leq c}{}^k$$



$$\leq \sum_{k=n+1}^{\infty} |a_k| \cdot c^k \leq B \cdot \sum_{k=n+1}^{\infty} q^k \xrightarrow{n \to \infty} 0$$

$\sum_{k=0}^{\infty} a_k \cdot \tilde{r}^k$ convergent for $c < \tilde{r} < r$

Hence there is B with $|a_k \tilde{r}^k| \leq B$

$B \geq |a_k| \cdot \tilde{r}^k = |a_k| \cdot c^k \cdot \Big(\underbrace{\tfrac{\tilde{r}}{c}}_{q^{-1}}\Big)^k$

(2) radius of convergence for $\sum_{k=1}^{\infty} \underbrace{a_k \cdot k}_{b_{k-1}} \cdot z^{k-1}$:

$$\limsup_{k \to \infty} \sqrt[k]{|a_{k+1}| \cdot (k+1)} = r^{-1}$$

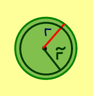
same proof as in (1)

(3) $\tilde{f}(z) := \sum_{k=1}^{\infty} a_k \cdot k \cdot z^{k-1}$, $p_N(z) := \sum_{k=0}^{N} a_k \cdot z^k$, $q_N(z) := \sum_{k=N+1}^{\infty} a_k \cdot z^k$

$$\left| \frac{f(z+h) - f(z)}{h} - \tilde{f}(z) \right| = \left| \frac{(p_N + q_N)(z+h) - (p_N + q_N)(z)}{h} - \tilde{f}(z) \right|$$

$$\leq \underbrace{\left| \frac{p_N(z+h) - p_N(z)}{h} - p_N'(z) \right|}_{A \xrightarrow{h \to 0} 0} + \underbrace{\left| p_N'(z) - \tilde{f}(z) \right|}_{B \xrightarrow{N \to \infty} 0} + \underbrace{\left| \frac{q_N(z+h) - q_N(z)}{h} \right|}_{C \xrightarrow{N \to \infty} 0}$$

For C: $$\left| \frac{\sum_{k=N+1}^{\infty} a_k (z+h)^k - \sum_{k=N+1}^{\infty} a_k \cdot z^k}{h} \right| \leq \sum_{k=N+1}^{\infty} |a_k| \underbrace{\left| \frac{(z+h)^k - z^k}{h} \right|}_{= (z+h)^{k-1} \cdot z^0 + (z+h)^{k-2} \cdot z^1 + \cdots + (z+h)^0 \cdot z^{k-1}}$$



$$\leq \sum_{k=N+1}^{\infty} |a_k| \cdot \tilde{r}^{k-1} \cdot k \xrightarrow{N \to \infty} 0$$

Geometric sum formula:

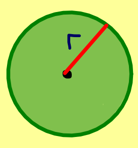
$$\frac{1 - q^k}{1 - q} = \sum_{j=0}^{k-1} q^j$$

Choose: $q = \frac{z}{z+h}$

Complex Analysis – Part 12

$$f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$$

→ holomorphic on its open disc of convergence



→ f' exists and is a power series

f'' exists and is a power series

$\vdots$

Examples: (1) $$\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad (\text{radius of convergence: } r = \infty)$$

$$\exp'(z) = \sum_{k=1}^{\infty} \frac{k \cdot z^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = \exp(z)$$

(2) $$\cos(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2 \cdot k}}{(2 \cdot k)!}$$

connection?

$$\exp(i \cdot z) = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!}, \qquad (iz)^k = \begin{cases} z^k, & k = 0, 4, 8, \dots \\ i z^k, & k = 1, 5, 9, 13, \dots \\ -z^k, & k = 2, 6, 10, \dots \\ -i z^k, & k = 3, 7, 11, \dots \end{cases}$$

$$\exp(-i \cdot z) = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!}, \qquad (-iz)^k \approx \begin{cases} z^k, & k = 0, 4, 8, \dots \\ -i z^k, & k = 1, 5, 9, 13, \dots \\ -z^k, & k = 2, 6, 10, \dots \\ +i z^k, & k = 3, 7, 11, \dots \end{cases}$$

$$\exp(i \cdot z) + \exp(-i \cdot z) = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!} \cdot 2 = 2 \cdot \cos(z)$$

$$\Longrightarrow \quad \cos(z) = \frac{1}{2} \Big(\exp(i \cdot z) + \exp(-i \cdot z) \Big)$$

$$\Longrightarrow \quad \cos'(z) = \frac{i}{2} \Big(\exp(i \cdot z) - \exp(-i \cdot z) \Big) = -\sin(z)$$

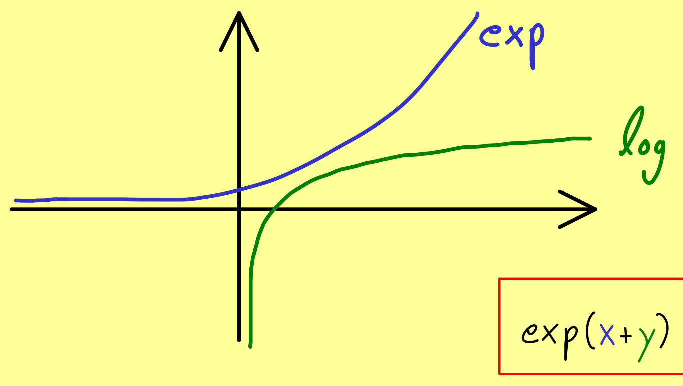
Complex Analysis – Part 13

logarithm $\log$ = inverse function of $\exp$

In $\mathbb{R}$:

$\exp : \mathbb{R} \longrightarrow (0,\infty)$

$\log : (0,\infty) \longrightarrow \mathbb{R}$



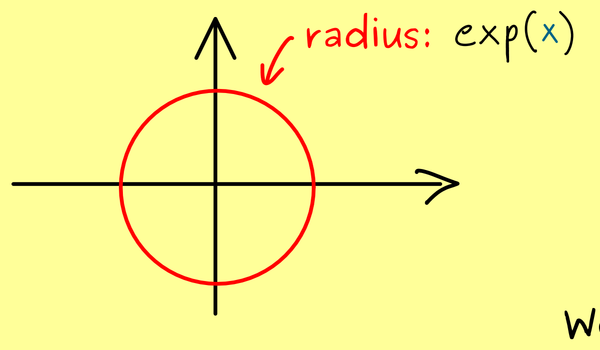
$\boxed{\exp(x+y) = \exp(x)\cdot\exp(y)}$

In $\mathbb{C}$: $z = x + iy$, $\exp(z) = \exp(x+iy)$

$= \exp(x)\cdot \underbrace{\exp(iy)}_{\cos(y) + i\sin(y)}$

Euler's formula

$$\boxed{\begin{aligned} |\exp(iy)|^2 &= \exp(iy)\exp(-iy) \\ &= \exp(iy - iy) \\ &= \exp(0) = 1 \end{aligned}}$$



define: $\frac{\pi}{2} :=$ smallest positive zero of $\cos : \mathbb{R} \to \mathbb{R}$

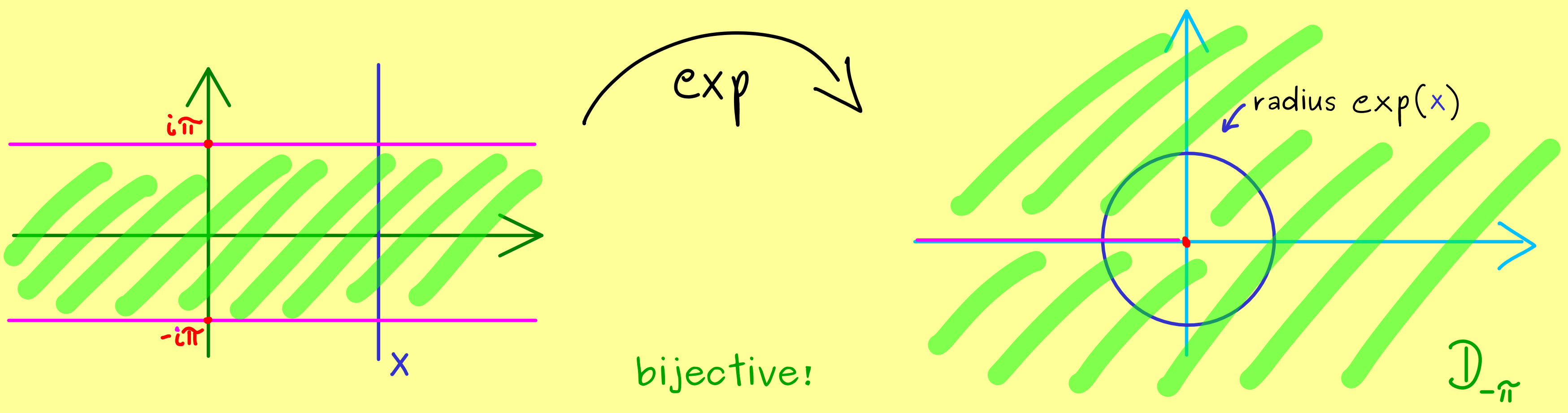
We get: $\exp\left(i\cdot\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

$= i\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = i$ (use derivative/monotonicity)

$\exp(i\cdot\pi) = -1$ and $\exp(i\cdot 2\pi) = 1$

Periodicity: $\exp(z + 2\pi i\cdot k) = \exp(z)$ for all $k \in \mathbb{Z}$, $z \in \mathbb{C}$

↳ not injective



Definition: $\log : D_{-\pi} \longrightarrow$ stripe is called the principal value of the logarithm function.

Properties: $\log(r\exp(i\varphi)) = \log(r) + i\varphi$, $\varphi \in (-\pi, \pi)$

$\log(\exp(i\varphi)) \xrightarrow{\varphi \to \pi} i\pi$

$\log(\exp(i\varphi)) \xrightarrow{\varphi \to -\pi} -i\pi$

$2\pi i$ jump

Complex Analysis – Part 14

$i^4 = \underbrace{i \cdot i \cdot i \cdot i}_{4 \text{ times}}$, $4^i = ?$

$\boxed{\log(x \cdot y) = \log(x) + \log(y)}$

Power definition in $\mathbb{R}$: $a > 0$, $m, n \in \mathbb{Z} \setminus \{0\}$ $\quad a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \exp\left(\log\left(\left(a^{\frac{1}{n}}\right)^m\right)\right)$

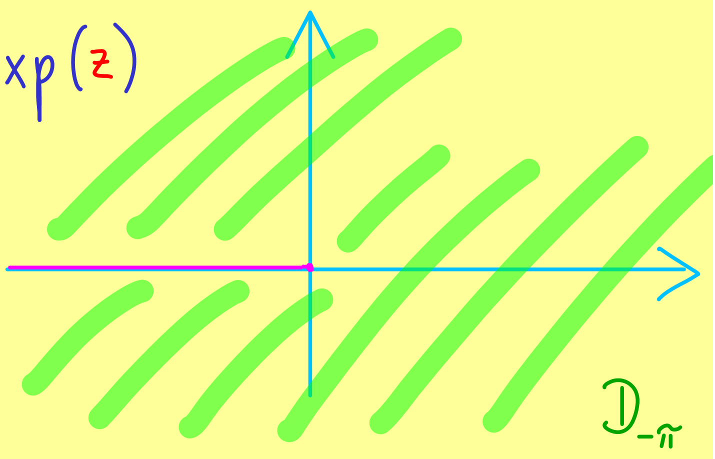
$$= \exp\left(m \cdot \log\left(a^{\frac{1}{n}}\right)\right)$$

$$= \exp\left(\tfrac{m}{n} \cdot \log(a)\right)$$

$a > 0$, $x \in \mathbb{R}$: $\quad a^x := \exp(x \cdot \log(a))$

Power definition in $\mathbb{C}$: $a > 0$, $z \in \mathbb{C}$: $\quad a^z := \exp(z \cdot \log(a))$

for example: $e^z = \exp(z)$



$\mathcal{D}_{-\pi}$

for complex base? $a \in \mathcal{D}_{-\pi}$, $z \in \mathbb{C}$:

$$a^z := \exp(z \cdot \log(a))$$

principal value of the logarithm

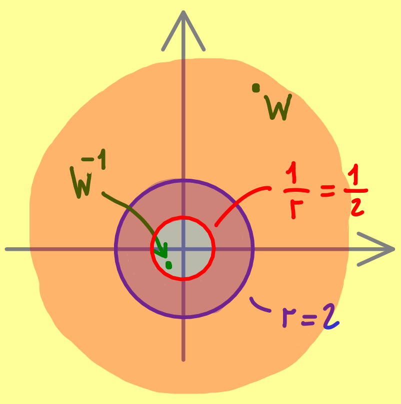
principal value of the power

be careful in calculations: $a^{z_1} \cdot a^{z_2} = a^{z_1 + z_2}$ ✓ $\qquad \left(a^{z_1}\right)^{z_2} \underset{\text{in general}}{\neq} a^{z_1 \cdot z_2}$

Complex Analysis – Part 15

Laurent series (generalisation of power series + holomorphic)

$\sum_{k=0}^{\infty} a_k \cdot z^k$ with radius of convergence $r \in [0, \infty]$



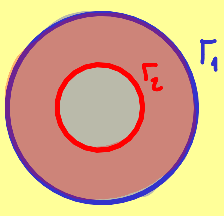
$\sum_{k=0}^{\infty} a_k \cdot \left(\frac{1}{w}\right)^k$ is convergent $\begin{cases} \left|\frac{1}{w}\right| < r \\ \Leftrightarrow \\ |w| > \frac{1}{r} \end{cases}$

chain rule

$\Rightarrow \quad w \mapsto \sum_{k=0}^{\infty} a_k \cdot w^{-k}$ is holomorphic on $\mathbb{C} \setminus \overline{B_{\frac{1}{r}}(0)}$

(alternatively: constant $+ \sum_{k=-1}^{-\infty} b_k \cdot z^k$)

Combine two series:



$z \mapsto \sum_{k=0}^{\infty} a_k \cdot z^k \rightsquigarrow$ with radius of convergence r_1

$\sum_{k=1}^{\infty} b_{-k} \cdot z^k \rightsquigarrow$ with radius of convergence r

$z \mapsto \sum_{k=-1}^{-\infty} b_k \cdot z^k \rightsquigarrow$ with "radius of convergence" $r_2 = \frac{1}{r}$

Definition: A Laurent series written as $\sum_{k=-\infty}^{\infty} a_k \cdot (z - z_0)^k$ is a pair of two series:

$z \mapsto \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k$ with radius of convergence $r_1 \in [0, \infty]$

principal part

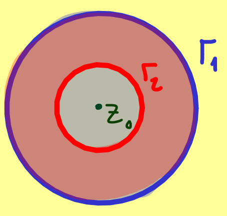
$\hookrightarrow z \mapsto \sum_{k=-1}^{-\infty} a_k \cdot (z - z_0)^k$ with "radius of convergence" $r_2 \in [0, \infty]$

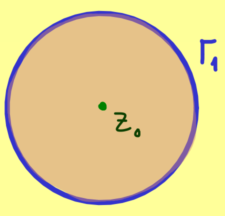
a_{-1} is called the residue of the Laurent series.

The Laurent series is a holomorphic function on $\{ z \in \mathbb{C} \mid r_2 < |z - z_0| < r_1 \}$

The Bright Side of Mathematics

Complex Analysis – Part 16

Laurent series: $\sum_{k=-\infty}^{\infty} a_k \cdot (z-z_0)^k$ with domain 

If the principal part is finite:  ← pole at z_0

<u>Definition:</u> Let $f: U \to \mathbb{C}$ be given by a Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k \cdot (z-z_0)^k$.

If there is $N \in \{-1,-2,\dots\}$ such that $a_k = 0$ for all $k < N$

and $a_N \neq 0$, then we say f has a pole of order $|N|$ at z_0.

<u>Example:</u> (a) $f(z) = \frac{1}{z}$ ← Laurent series $\sum_{k=-\infty}^{\infty} a_k \cdot z^k$

$\Rightarrow$ f has a pole of order 1 at 0.

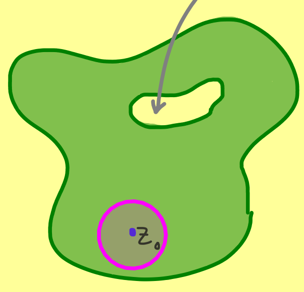
(b) $f(z) = \frac{1}{z} + \frac{1}{z^2}$ $\Rightarrow$ f has a pole of order 2 at 0.

<u>Definition:</u> Let $f: U \to \mathbb{C}$ be holomorphic and $z_0 \notin U$.

If there is $\varepsilon > 0$ with $B_\varepsilon(z_0) \setminus \{z_0\} \subseteq U$,

then z_0 is called an <u>isolated singularity</u> of f.

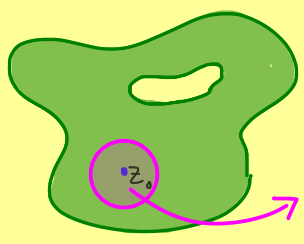
not isolated



<u>Example:</u> $f(z) = \frac{1}{z(z-1)}$ is holomorphic with domain $\mathbb{C} \setminus \{0, 1\}$

isolated singularities

<u>Proposition:</u> At isolated singularities, we always find a Laurent series locally:



proof later

$$B_\varepsilon(z_0) \setminus \{z_0\} \ni z \longmapsto \sum_{k=-\infty}^{\infty} a_k \cdot (z-z_0)^k = f(z)$$

uniquely given

<u>Three cases for isolated singularities:</u>

(1) removable singularity: $\forall k < 0 : \; a_k = 0$

(2) pole: $\exists N \in \{-1,-2,\dots\} \; \forall k < N : \; a_k = 0$ and $a_N \neq 0$

(3) essential singularity: $\forall N \in \{-1,-2,\dots\} \; \exists k \leq N \; a_k \neq 0$

<u>Examples:</u> (1) $f(z) = \frac{\sin(z)}{z} = \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!}$ $\quad z_0 = 0$ removable singularity

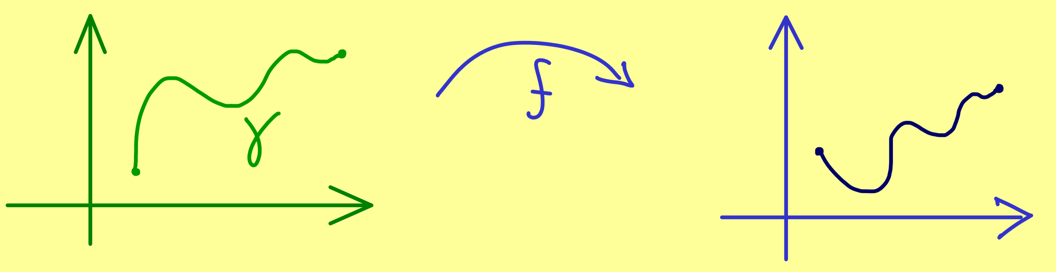
(2) $f(z) = \frac{\sin(z)}{z^2} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-1}}{(2k+1)!}$ $\quad z_0 = 0$ pole of order 1

(3) $f(z) = \exp\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}$ $\quad z_0 = 0$ essential singularity

Complex Analysis – Part 17

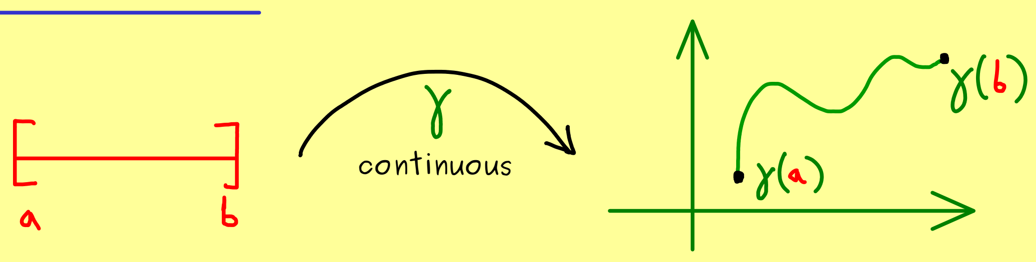
Complex integration: $f: \mathbb{C} \to \mathbb{C}$

↳ curve integral, line integral, contour integral



$$\int_\gamma f(z)\, dz$$

Complex integration on real intervals:



For a continuous map $\gamma: [a,b] \longrightarrow \mathbb{C}$, we define:

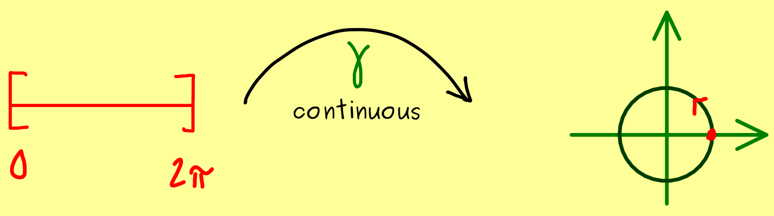
$$\int_a^b \gamma(t)\, dt := \int_a^b \mathrm{Re}(\gamma(t))\, dt + i \cdot \int_a^b \mathrm{Im}(\gamma(t))\, dt$$

ordinary Riemann integrals in $\mathbb{R}$

Important property: Let $\gamma: [a,b] \longrightarrow \mathbb{C}$ be continuous. Then:

$$\left| \int_a^b \gamma(t)\, dt \right| \leq \int_a^b |\gamma(t)|\, dt$$

Example: $\gamma: [0, 2\pi] \longrightarrow \mathbb{C}$, $\gamma(t) = e^{it}$



$$\int_a^b e^{it}\, dt = \int_a^b \cos(t)\, dt + i \cdot \int_a^b \sin(t)\, dt$$

$$= \sin(t)\Big|_a^b + i \cdot \left(-\cos(t)\right)\Big|_a^b = -i \cdot \cos(t) + \sin(t)\Big|_a^b$$

$$= \frac{1}{i}\left(\cos(t) + i \cdot \sin(t)\right)\Big|_a^b = \frac{1}{i} e^{it}\Big|_a^b$$

Proof of $\boxed{\left|\int_a^b \gamma(t)\, dt\right| \leq \int_a^b |\gamma(t)|\, dt}$: Assume $0 \neq \underbrace{\int_a^b \gamma(t)\, dt}_{=: w} \in \mathbb{C}$. Define: $c := \frac{w}{|w|}$. Then:

$$\int_a^b \mathrm{Re}(\bar{c}^{-1} \cdot \gamma(t))\, dt = \int_a^b \bar{c}^{-1} \cdot \gamma(t)\, dt = \bar{c}^{-1} \int_a^b \gamma(t)\, dt = |w| \in \mathbb{R}$$

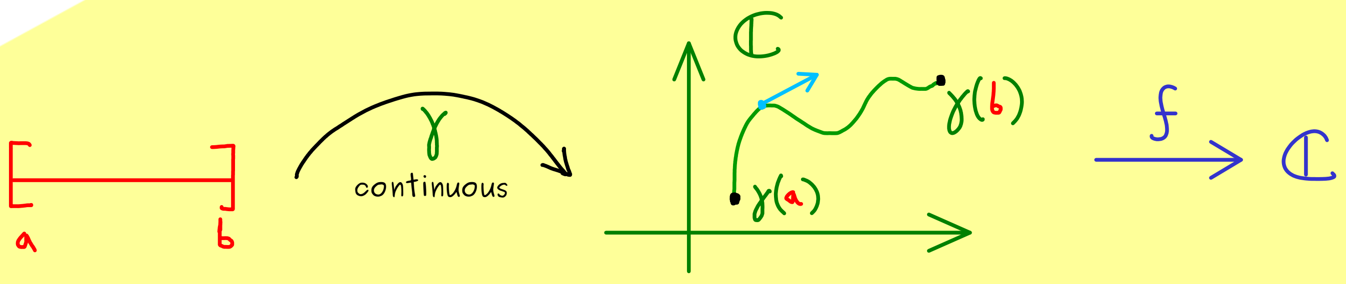
We know: $\left|\mathrm{Re}(\bar{c}^{-1} \cdot \gamma(t))\right| \leq \left|\bar{c}^{-1} \cdot \gamma(t)\right| = \underbrace{|\bar{c}^{-1}|}_{=1} \cdot |\gamma(t)|$

$$\Rightarrow \int_a^b \left|\mathrm{Re}(\bar{c}^{-1} \cdot \gamma(t))\right| dt \leq \int_a^b |\gamma(t)|\, dt$$

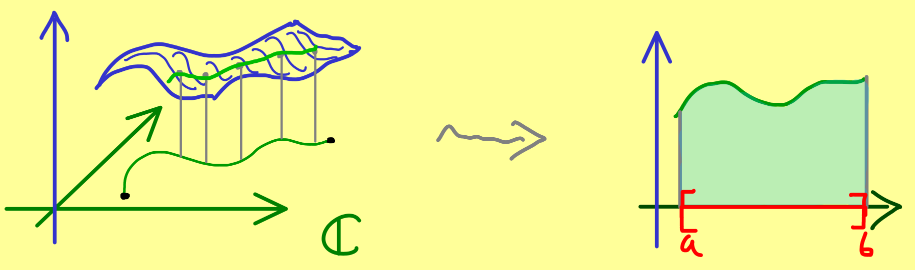
$$\mathbin{/\!/}$$

$$\left| \int_a^b \gamma(t)\, dt \right| = \left| \int_a^b \mathrm{Re}(\bar{c}^{-1} \cdot \gamma(t))\, dt \right|$$

Complex Analysis – Part 18



First look at $f: \mathbb{C} \longrightarrow \mathbb{R}$

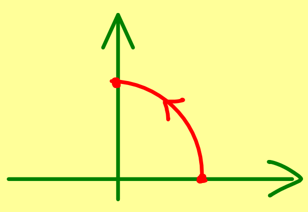


Definition: For a parametrized curve $\gamma: [a,b] \longrightarrow \mathbb{C}$ continuously differentiable with $\gamma': [a,b] \longrightarrow \mathbb{C}$, we define:

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

for continuous functions $f: U \longrightarrow \mathbb{C}$ with $\mathrm{Ran}(\gamma) \subseteq U$.

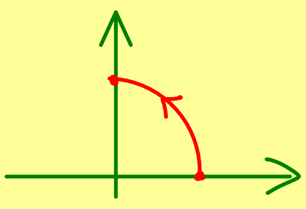
Examples: (a) $f(z) = z$, $\gamma_1: [0, \frac{\pi}{2}] \longrightarrow \mathbb{C}$, $t \longmapsto e^{it}$



$$\int_{\gamma_1} f(z)\, dz = \int_0^{\frac{\pi}{2}} f(\underbrace{\gamma_1(t)}_{e^{it}}) \cdot \underbrace{\gamma_1'(t)}_{i \cdot e^{it}}\, dt = i \cdot \int_0^{\frac{\pi}{2}} e^{2it}\, dt = i \cdot \frac{1}{2i} e^{2it} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \cdot \left(e^{i\pi} - 1\right) = -1$$

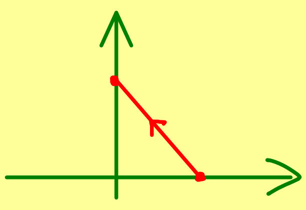
(b) $f(z) = z$, $\gamma_2: [0,1] \longrightarrow \mathbb{C}$, $t \longmapsto e^{i\frac{\pi}{2}t}$



$$\int_{\gamma_2} f(z)\, dz = \int_0^1 f(\underbrace{\gamma_2(t)}_{e^{i\frac{\pi}{2}t}}) \cdot \underbrace{\gamma_2'(t)}_{i\frac{\pi}{2}e^{i\frac{\pi}{2}t}}\, dt = i \cdot \frac{\pi}{2} \int_0^1 e^{i\pi t}\, dt = i\frac{\pi}{2} \frac{1}{i\pi} e^{i\pi t} \Big|_0^1$$

$$= -1$$

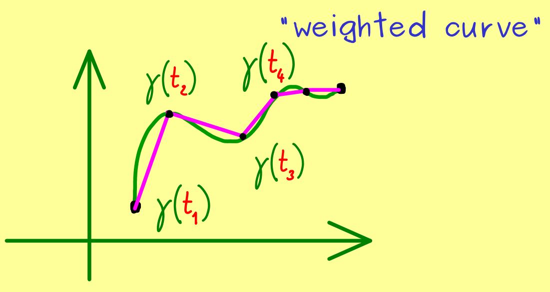
(c) $f(z) = z$, $\gamma_3: [0,1] \longrightarrow \mathbb{C}$, $t \longmapsto (1-t) + i \cdot t$



$$\int_{\gamma_3} f(z)\, dz = \int_0^1 f(\gamma_3(t)) \cdot \underbrace{\gamma_3'(t)}_{-1+i}\, dt = (-1+i) \int_0^1 \left(1 + (i-1)t\right) dt$$

$$= (-1+i)\left(t + \frac{1}{2}(i-1)t^2\right)\Big|_0^1 = (-1+i)\left(1 + \frac{1}{2}(i-1)\right) = -1$$

Another visualisation:



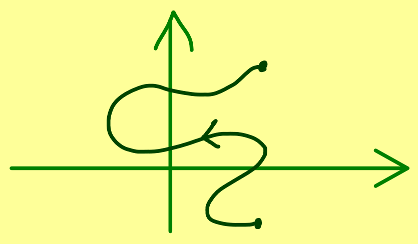
$$\sum_{i=1}^n f(\gamma(t_i)) \cdot \left(\gamma(t_{i+1}) - \gamma(t_i)\right)$$

$$= \sum_{i=1}^n f(\gamma(t_i)) \frac{\gamma(t_{i+1}) - \gamma(t_i)}{t_{i+1} - t_i} (t_{i+1} - t_i)$$

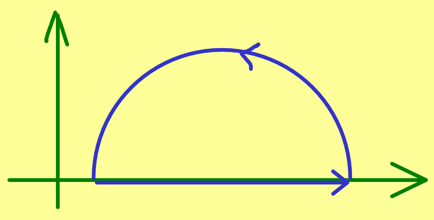
$$\xrightarrow[\text{(in some sense)}]{n \to \infty} \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

Complex Analysis – Part 19

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$



$\gamma : [a,b] \to \mathbb{C}$ continuously differentiable



We can extend this: $\gamma : [a,b] \to \mathbb{C}$ piecewise continuously differentiable

there are $\underset{a}{\overset{}{a_1}}\!\!{}^{\shortparallel}, a_2, a_3, \dots, \underset{=b}{a_{n+1}} \in [a,b]$

such that $\gamma|_{[a_i, a_{i+1}]}$ is continuously differentiable

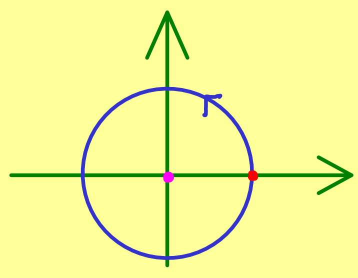
define: $$\int_\gamma f(z)\,dz := \sum_{i=1}^{n} \int_{\gamma|_{[a_i,a_{i+1}]}} f(z)\,dz$$

If $\gamma(a) = \gamma(b)$, then γ is called a closed curve and we write:

$$\oint_\gamma f(z)\,dz$$

Important example: $\oint_\gamma \frac{1}{z}\,dz$, $\gamma : [0, 2\pi] \to \mathbb{C}$, $t \mapsto e^{it}$

$$= \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot i\,\cancel{e^{it}}\,dt = 2\pi \cdot i$$

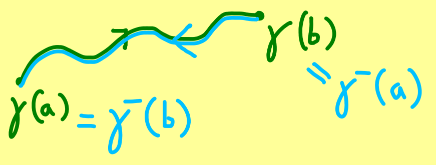


Properties: $f, g : U \to \mathbb{C}$ continuous, $\gamma : [a,b] \to \mathbb{C}$ piecewise continuously differentiable

(a) $\int_\gamma \left(\alpha \cdot f(z) + \beta \cdot g(z)\right) dz = \alpha \cdot \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz$ for all $\alpha, \beta \in \mathbb{C}$

(b) If γ^- is γ with reverse orientation,

$$\left(\gamma^-(t) := \gamma(-t + a + b)\right)$$



then $\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz$

(c) $$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt\right| \le \int_a^b \left|f(\gamma(t))\cdot\gamma'(t)\right| dt$$

$$= \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\,dt \le \sup_{z \in \mathrm{Ran}(\gamma)} |f(z)| \cdot \int_a^b |\gamma'(t)|\,dt$$

$$= \max_{z \in \mathrm{Ran}(\gamma)} |f(z)| \cdot \mathrm{length}(\gamma)$$

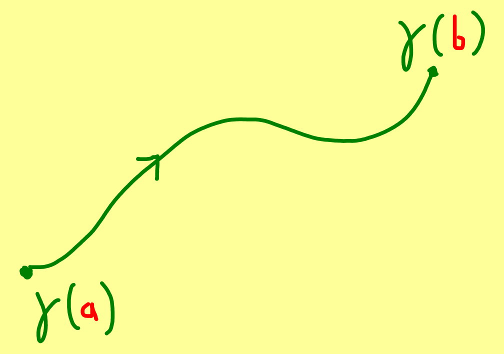
Complex Analysis – Part 20

Definition: $U \subseteq \mathbb{C}$ open, $f: U \to \mathbb{C}$.

$F: U \to \mathbb{C}$ is called a primitive/antiderivative of f

if $F' = f$. complex derivative!

Fact: If $f: U \to \mathbb{C}$ has an antiderivative $F: U \to \mathbb{C}$, then:



$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

parametrized curve $\gamma: [a,b] \to U$

Proof:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt,$$

$$= \int_a^b \frac{d}{dt}(F \circ \gamma)(t)\,dt$$

$$\boxed{\frac{d}{dt}(F \circ \gamma)(t) = F'(\gamma(t)) \cdot \gamma'(t) \quad \text{chain rule}}$$

fundamental theorem of calculus

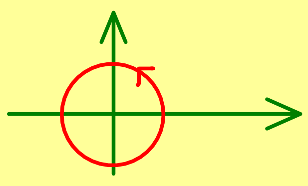
$$= (F \circ \gamma)(t)\Big|_a^b = F(\gamma(b)) - F(\gamma(a))$$

Corollary: If $f: U \to \mathbb{C}$ has an antiderivative and γ is closed, then:

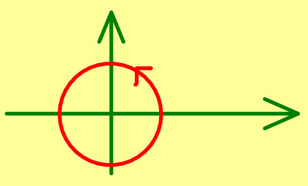
$$\oint_\gamma f(z)\,dz = 0$$

Example: (a) $U = \mathbb{C}\setminus\{0\}$, $f(z) = \frac{1}{z^2}$ antiderivative: $F(z) = -\frac{1}{z}$

$$\Rightarrow \oint_\gamma f(z)\,dz = 0$$



(b) $U = \mathbb{C}\setminus\{0\}$, $f(z) = \frac{1}{z}$

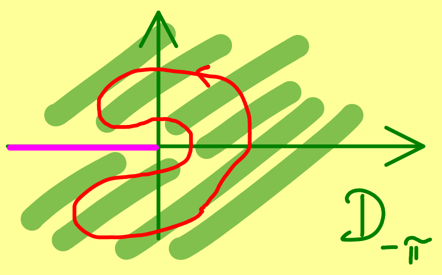


We know: $\oint_\gamma f(z)\,dz = 2\pi i$ with $\gamma: [0,2\pi] \to U$, $\gamma(t) = e^{it}$

$\Rightarrow$ no antiderivative for $\frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$

(c) $U = D_{-\pi}$

$\log: D_{-\pi} \to \mathbb{C}$



$\log'(z) = \frac{1}{z} \quad \Rightarrow \oint_\gamma f(z)\,dz = 0$

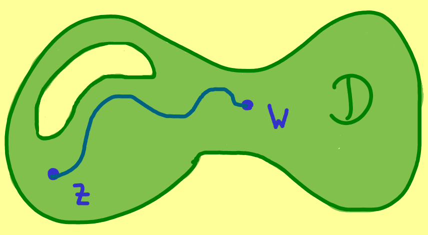
Complex Analysis – Part 21

Fact: $f: U \to \mathbb{C}$ has an antiderivative $\overset{?}{\Longleftrightarrow}$ $\oint_\gamma f(z)\,dz = 0$ for all closed curves γ

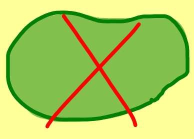
Theorem: $f: \underset{\smile}{D} \to \mathbb{C}$ holomorphic

open domain/region: open + $\underbrace{\text{path-connected}}$

for any two points $z, w \in D$
there is a curve $\gamma: [a,b] \to D$ with $\gamma(a) = z$ and $\gamma(b) = w$

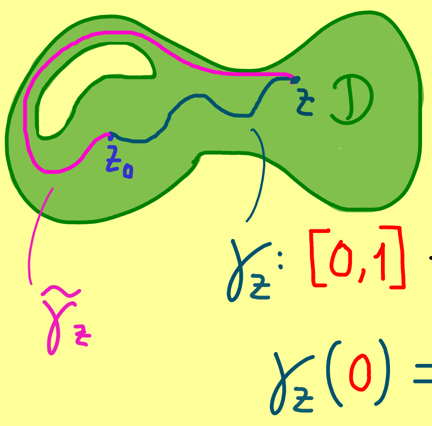


not allowed:



If $\oint_\gamma f(z)\,dz = 0$ for all closed curves γ, then f has an antiderivative.

Proof:



$\gamma_z: [0,1] \to D$

$\gamma_z(0) = z_0, \quad \gamma_z(1) = z$

For $z_0, z \in D$, define:

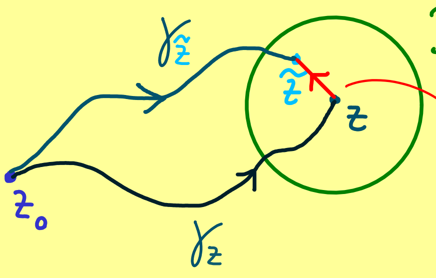
$$F(z) := \int_{\gamma_z} f(\zeta)\,d\zeta$$

well-defined!

$\Rightarrow$ $\tilde{\gamma}_z^- + \gamma_z$ closed curve: $0 = \oint_{\tilde{\gamma}_z^- + \gamma_z} f(\zeta)\,d\zeta = \int_{\tilde{\gamma}_z^-} f(\zeta)\,d\zeta + \int_{\gamma_z} f(\zeta)\,d\zeta$

$$\Longrightarrow \int_{\gamma_z} f(\zeta)\,d\zeta = \int_{\tilde{\gamma}_z} f(\zeta)\,d\zeta$$

Show: $F' = f$



$\gamma_{z\tilde{z}}$ line connecting $\tilde{z}$ with z

$$\left| \frac{F(\tilde{z}) - F(z)}{\tilde{z} - z} - f(z) \right| = \left| \frac{F(\tilde{z}) - F(z) - f(z)(\tilde{z} - z)}{\tilde{z} - z} \right|$$

$$= \frac{1}{|\tilde{z} - z|} \cdot \left| \int_{\gamma_{\tilde{z}}} f(\zeta)\,d\zeta - \int_{\gamma_z} f(\zeta)\,d\zeta - f(z)(\tilde{z} - z) \right|$$

$$= \frac{1}{|\tilde{z} - z|} \cdot \left| \int_{\gamma_{z\tilde{z}}} f(\zeta)\,d\zeta - \int_{\gamma_{z\tilde{z}}} f(z)\,d\zeta \right| = \frac{1}{|\tilde{z} - z|} \cdot \left| \int_{\gamma_{z\tilde{z}}} \big(f(\zeta) - f(z)\big)\,d\zeta \right|$$

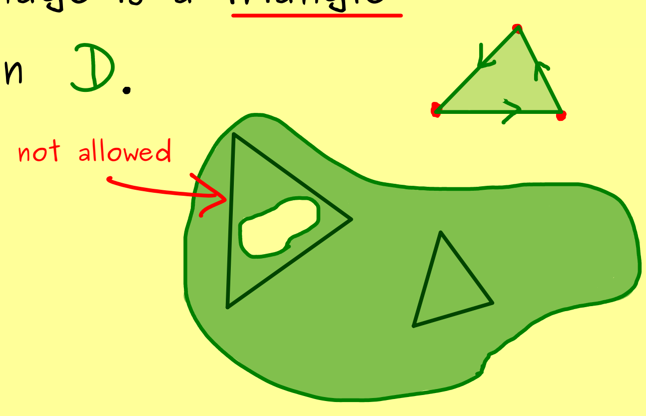
$$\leq \frac{1}{\cancel{|\tilde{z} - z|}} \max_{\zeta \in \mathrm{Ran}(\gamma_{z\tilde{z}})} |f(\zeta) - f(z)| \cdot \cancel{\mathrm{length}}(\gamma_{z\tilde{z}}) \leq \max_{\zeta \in B_\varepsilon(z)} |f(\zeta) - f(z)| \xrightarrow{\varepsilon \to 0} 0$$

The Bright Side of Mathematics

Complex Analysis – Part 22

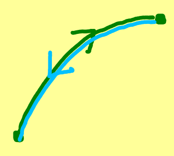
Goursat's theorem: $f: D \to \mathbb{C}$ holomorphic,

$\gamma: [a,b] \longrightarrow D$ closed curve where the image is a triangle and the inner part lies in D.

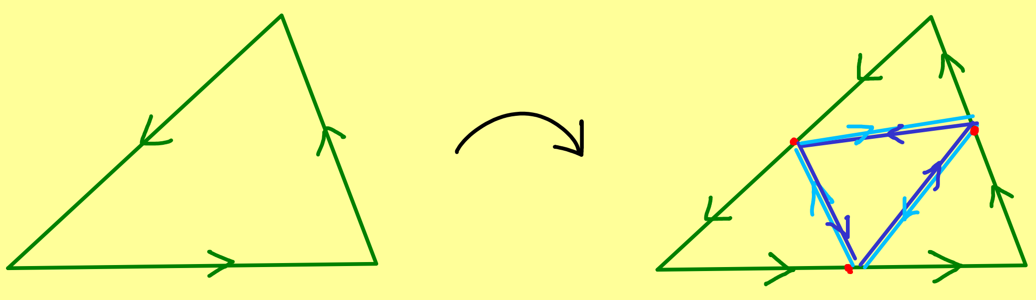


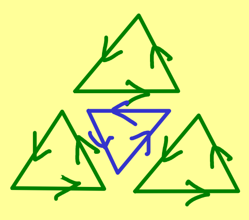
Then: $$\oint_\gamma f(z)\,dz = 0$$

Proof:

Basic idea:  $$0 = \int_{\nearrow} + \int_{\swarrow} = \oint_{\nearrow + \swarrow}$$

Decompose triangle:





$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$$\oint_\gamma f(z)\,dz = \oint_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)\,dz$$

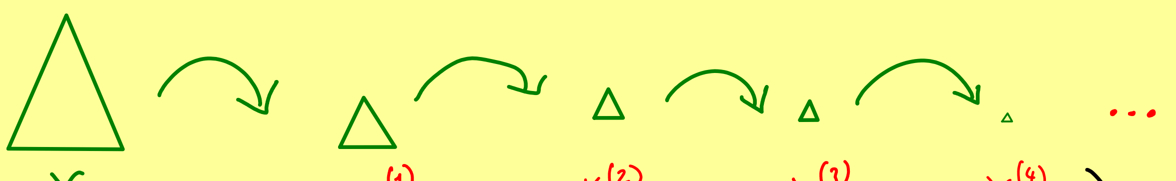
$$= \oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz + \oint_{\gamma_3} f(z)\,dz + \oint_{\gamma_4} f(z)\,dz$$

$$\left|\oint_\gamma f(z)\,dz\right| \leq \left|\oint_{\gamma_j} f(z)\,dz\right| + \left|\oint_{\gamma_j} f(z)\,dz\right| + \left|\oint_{\gamma_j} f(z)\,dz\right| + \left|\oint_{\gamma_j} f(z)\,dz\right|$$

$$= 4 \cdot \left|\oint_{\gamma^{(1)}} f(z)\,dz\right|$$

γ_j represents maximal value $= \gamma^{(1)}$

Repeat n times:



$\gamma \quad \gamma^{(1)} \quad \gamma^{(2)} \quad \gamma^{(3)} \quad \gamma^{(4)} \quad \dots$

converge to a point $n \to \infty$

z_0 lies in the interior of all $\gamma^{(n)}$

$$\left|\oint_\gamma f(z)\,dz\right| \leq 4^n \cdot \left|\oint_{\gamma^{(n)}} f(z)\,dz\right| \qquad (*)$$

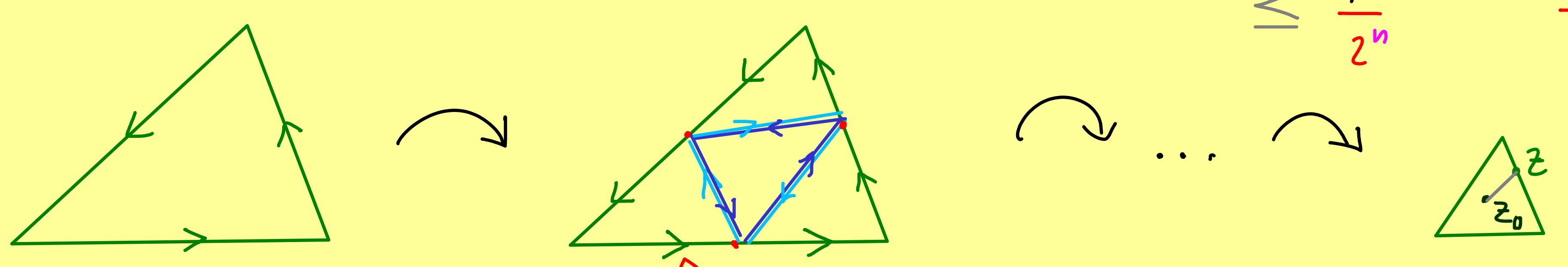
Complex differentiability at z_0:

$$f(z) = \underbrace{f(z_0) + f'(z_0)\cdot(z-z_0)}_{\text{has antiderivative } \Rightarrow \oint = 0} + \varphi(z) \qquad \text{where } \frac{\varphi(z)}{z-z_0} \xrightarrow{z \to z_0} 0$$

$$\varphi(z) = \psi(z)(z-z_0) \quad \text{with } \psi(z) \xrightarrow{z \to z_0} 0$$

$$\left|\oint_{\gamma^{(n)}} f(z)\,dz\right| = \left|\oint_{\gamma^{(n)}} \varphi(z)\,dz\right| \leq \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\varphi(z)| \cdot \mathrm{length}(\gamma^{(n)})$$

$$\leq \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\psi(z)| \cdot \underbrace{\max_{z \in \mathrm{Ran}(\gamma^{(n)})} |z-z_0|}_{\leq \frac{r}{2^n}} \cdot \underbrace{\mathrm{length}(\gamma^{(n)})}_{\frac{r}{2^n}}$$



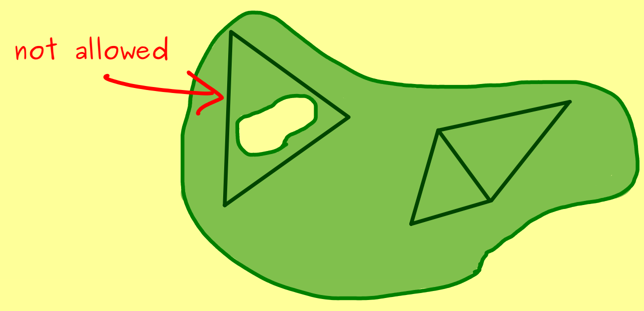
γ length $= r$ $\qquad \gamma^{(1)}$ length $\frac{r}{2}$ $\qquad \gamma^{(n)}$ length $\frac{r}{2^n}$

$$\left|\oint_\gamma f(z)\,dz\right| \overset{(*)}{\leq} 4^n \cdot \left|\oint_{\gamma^{(n)}} f(z)\,dz\right| \leq r^2 \cdot \max_{z \in \mathrm{Ran}(\gamma^{(n)})} |\psi(z)| \xrightarrow{n \to \infty} 0$$

$\square$

Complex Analysis – Part 23

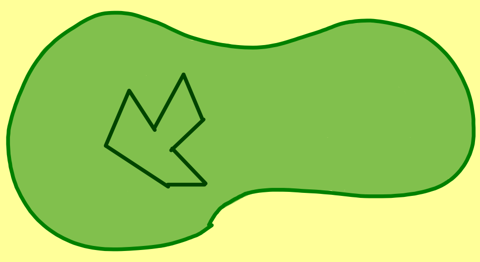
$f: D \to \mathbb{C}$ holomorphic



$\triangle \subseteq D \Rightarrow \oint_{\triangle} f(z)\, dz = 0$

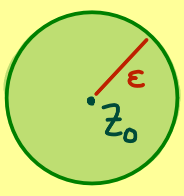
$\subseteq D \Rightarrow \oint f(z)\, dz = 0$

$\subseteq D \Rightarrow \oint f(z)\, dz = 0$

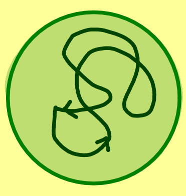


Cauchy's theorem (for a disc):

$f: D \to \mathbb{C}$ holomorphic where $D = B_\varepsilon(z_0)$ (open disc)



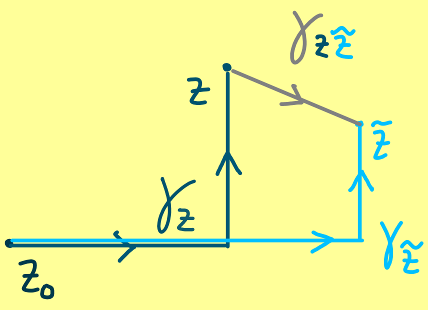
$\gamma: [a,b] \to D$ closed curve



$$\Rightarrow \oint_\gamma f(z)\, dz = 0$$

Proof: Show that an antiderivative exists!

$$F(\tilde{z}) := \int_{\gamma_{\tilde{z}}} f(\zeta)\, d\zeta$$



(everything lies in D)

$$\oint_{\gamma_{\tilde{z}} + \gamma_z^- + \gamma_{z\tilde{z}}^-} f(z)\, dz \overset{\text{Goursat}}{=} 0 \quad (*)$$

$$\left| \frac{F(\tilde{z}) - F(z)}{\tilde{z} - z} - f(z) \right| = \frac{1}{|\tilde{z} - z|} \cdot \left| \int_{\gamma_{\tilde{z}}} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta - f(z)(\tilde{z} - z) \right|$$

$$\overset{(*)}{=} \frac{1}{|\tilde{z} - z|} \cdot \left| \int_{\gamma_{z\tilde{z}}} \big(f(\zeta) - f(z) \big)\, d\zeta \right|$$

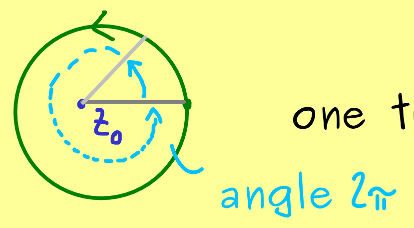
$$\leq \frac{1}{\cancel{|\tilde{z} - z|}} \max_{\zeta \in \operatorname{Ran}(\gamma_{z\tilde{z}})} |f(\zeta) - f(z)| \cdot \cancel{\operatorname{length}}(\gamma_{z\tilde{z}}) \xrightarrow{\tilde{z} \to z} 0$$

$\Rightarrow f$ has an antiderivative on D $\Rightarrow \oint_\gamma f(z)\, dz = 0$ for each closed curve γ in D

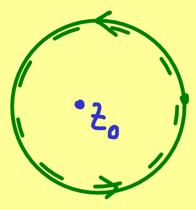
Complex Analysis – Part 24

Winding number for curves

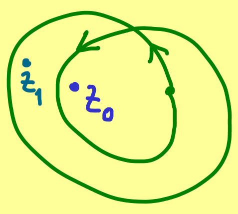
$\gamma : [a,b] \to \mathbb{C}$ piecewise continuously differentiable



one turn around z_0



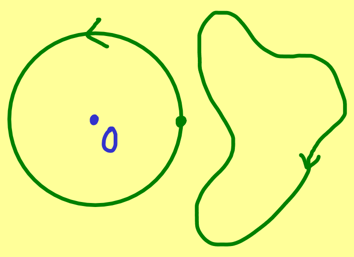
two turns around z_0



two turns around z_0

one turn around z_1

Special integral:



$$\oint_\gamma \frac{1}{z}\, dz = \begin{cases} 2\pi i\,, & \text{for } \gamma : [0,2\pi] \to \mathbb{C}\,,\ t \mapsto e^{it} \\ 4\pi i\,, & \text{for } \gamma : [0,4\pi] \to \mathbb{C}\,,\ t \mapsto e^{it} \\ 0\,, & \text{for } \gamma : [a,b] \to \mathbb{C} \text{ with image in a disk where } 0 \notin \text{disk} \end{cases}$$

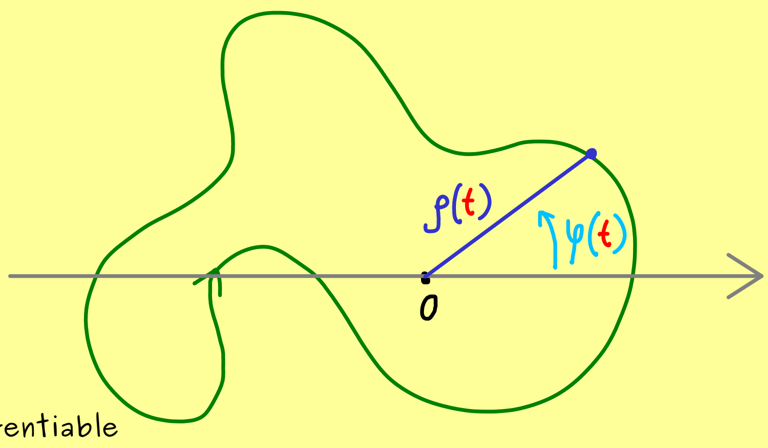
$$\Rightarrow \quad \frac{1}{2\pi i} \oint_\gamma \frac{1}{z}\, dz = 1 \quad \text{for } \gamma : [0,2\pi] \to \mathbb{C}\,,\ t \mapsto e^{it}$$

Definition: The winding number of a curve γ around $z_0 \in \mathbb{C}$ $(z_0 \notin \mathrm{Ran}(\gamma))$ is defined by:

$$\mathrm{wind}(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz$$

Fact: γ closed $\Rightarrow$ $\mathrm{wind}(\gamma, z_0) \in \mathbb{Z}$

Proof: Assume $z_0 = 0$, $\gamma : [a,b] \to \mathbb{C}$ closed



Write γ as: $\gamma(t) = \rho(t) \cdot e^{i\varphi(t)}$

piecewise continuously differentiable

$$\oint_\gamma \frac{1}{z}\, dz = \int_a^b \frac{1}{\gamma(t)} \gamma'(t)\, dt = \int_a^b \frac{1}{\rho(t) \cdot e^{i\varphi(t)}} \Big(\rho'(t) \cdot e^{i\varphi(t)} + \rho(t)\, i \varphi'(t)\, e^{i\varphi(t)} \Big) dt$$

$$= \int_a^b \frac{\rho'(t)}{\rho(t)}\, dt + i \int_a^b \varphi'(t)\, dt$$

$$= \log\big(\rho(t)\big)\Big|_a^b + i\, \varphi(t) \Big|_a^b$$

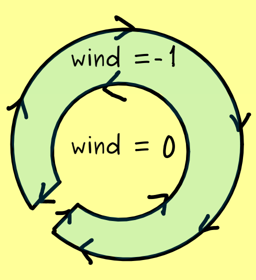
$$\boxed{\begin{aligned} &\gamma \text{ closed} \\ \Rightarrow\ &\rho(b) = \rho(a) \\ &\varphi(b) = \varphi(a) + 2\pi k \\ &k \in \mathbb{Z} \end{aligned}}$$

$$= 0 + i\, 2\pi k$$

$$\Rightarrow \quad \mathrm{wind}(\gamma, 0) = k$$

Complex Analysis – Part 25

winding number: $\text{wind}(\gamma, z_0) := \frac{1}{2\pi i} \oint_\gamma \frac{1}{z - z_0}\, dz$



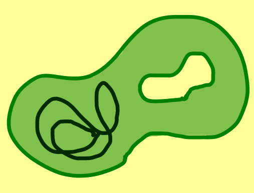
<u>Definition:</u> For $\gamma : [a,b] \to \mathbb{C}$ closed:

$$\text{Ext}(\gamma) := \{ z_0 \in \mathbb{C} \setminus \text{Ran}(\gamma) \mid \text{wind}(\gamma, z_0) = 0 \}$$

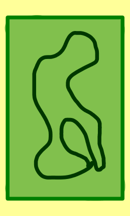
$$\text{Int}(\gamma) := \{ z_0 \in \mathbb{C} \setminus \text{Ran}(\gamma) \mid \text{wind}(\gamma, z_0) \neq 0 \}$$

<u>Extending Cauchy's theorem:</u>

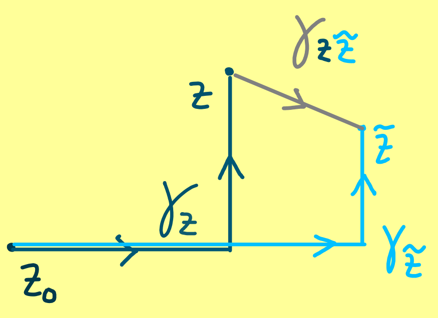
$f : D \to \mathbb{C}$ holomorphic, γ closed, $\text{Int}(\gamma) \cup \text{Ran}(\gamma) \subseteq D$



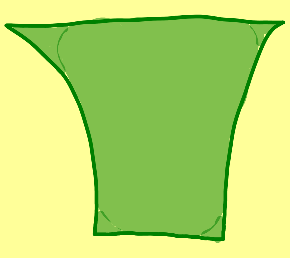
$D =$ disc  $\overset{\text{part 23}}{\Longrightarrow}$ $\oint_\gamma f(z)\, dz = 0$

$D =$ rectangle  $\overset{\text{same proof part 23}}{\Longrightarrow}$ $\oint_\gamma f(z)\, dz = 0$

proof needed: $F(z) := \int_{\gamma_z} f(\zeta)\, d\zeta$



$\oint_{\gamma_{\tilde z} + \gamma_z^- + \gamma_{z\tilde z}^-} f(z)\, dz \overset{\text{Goursat}}{=} 0$

works also: $D =$  , $D =$ 

<u>Cauchy's theorem (general version):</u>

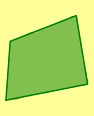
$f : D \to \mathbb{C}$ holomorphic, γ closed, $\text{Int}(\gamma) \cup \text{Ran}(\gamma) \subseteq D$

$$\Longrightarrow \oint_\gamma f(z)\, dz = 0$$

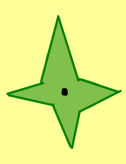
<u>Cauchy's theorem (for some domains):</u>

$f : D \to \mathbb{C}$ holomorphic, $\gamma : [a,b] \to D$ closed curve,

If $\left\{ \begin{array}{l} D \text{ convex} \\ D = \text{(ring)} \\ D \text{ star domains} \end{array} \right\}$ $\Longrightarrow \oint_\gamma f(z)\, dz = 0$

D convex  or

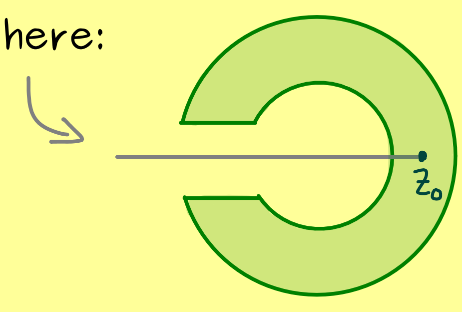
$D =$  or

D star domains 

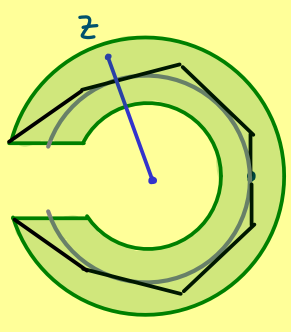
<u>Appendix:</u>

Proof from part 23 can be transformed to a proof for domain $D =$ 

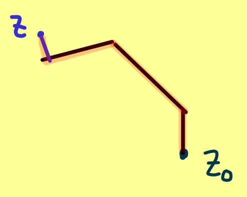
Just fix point z_0 here:



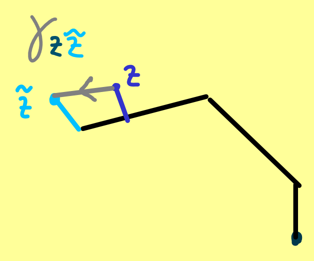
Then for every z, define the path γ_z in the following way:



So it's a well-defined polygon path:



Hence, we have $\oint_{\gamma_{\tilde z} + \gamma_z^- + \gamma_{z\tilde z}^-} f(z)\, dz \overset{\text{Goursat}}{=} 0$ (*) if $\tilde z$ and z are close enough.



So for $F(z) := \int_{\gamma_z} f(\zeta)\, d\zeta$, we also get:

$$\left| \frac{F(\tilde z) - F(z)}{\tilde z - z} - f(z) \right| = \frac{1}{|\tilde z - z|} \cdot \left| \int_{\gamma_{\tilde z}} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta - f(z)(\tilde z - z) \right|$$

$$\overset{(*)}{=} \frac{1}{|\tilde z - z|} \cdot \left| \int_{\gamma_{z\tilde z}} \big(f(\zeta) - f(z) \big)\, d\zeta \right|$$

$$\leq \frac{1}{\cancel{|\tilde z - z|}} \max_{\zeta \in \text{Ran}(\gamma_{z\tilde z})} |f(\zeta) - f(z)| \cdot \cancel{\text{length}(\gamma_{z\tilde z})} \xrightarrow{\tilde z \to z} 0$$

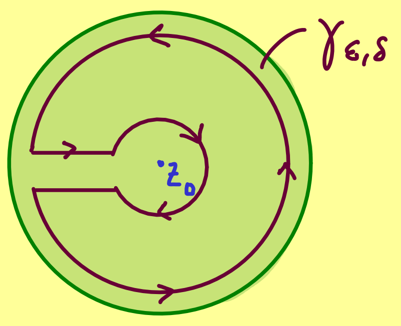
$\Longrightarrow$ f has an antiderivative on D $\Longrightarrow$ $\oint_\gamma f(z)\, dz = 0$ for each closed curve γ in D

Complex Analysis – Part 26

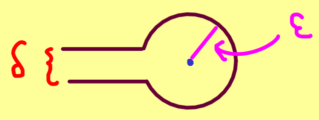
$\rightsquigarrow$ Cauchy's theorem applicable

$\rightsquigarrow$ keyhole contour

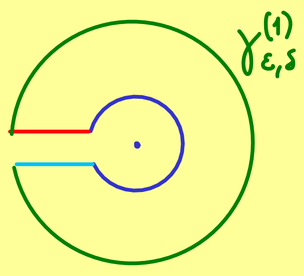
Assume: $g: B_r(z_0)\setminus\{z_0\} \longrightarrow \mathbb{C}$ holomorphic



$$\oint_{\gamma_{\varepsilon,\delta}} g(z)\,dz \overset{\text{Cauchy's theorem}}{=} 0$$



Split it up:

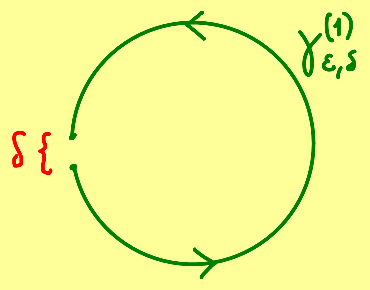


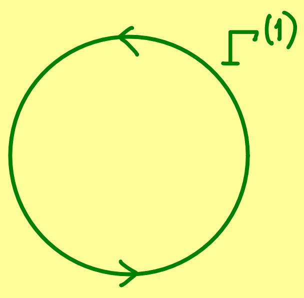
$$\gamma_{\varepsilon,\delta} = \gamma^{(1)}_{\varepsilon,\delta} + \gamma^{(2)}_{\varepsilon,\delta} + \gamma^{(3)}_{\varepsilon,\delta} + \gamma^{(4)}_{\varepsilon,\delta}$$

$$\Longrightarrow \int_{\gamma^{(1)}_{\varepsilon,\delta}} g(z)\,dz + \int_{\gamma^{(2)}_{\varepsilon,\delta}} g(z)\,dz + \int_{\gamma^{(4)}_{\varepsilon,\delta}} g(z)\,dz + \int_{\gamma^{(3)}_{\varepsilon,\delta}} g(z)\,dz = 0$$

What happens for $\delta \to 0$?

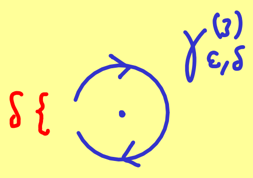
(1)



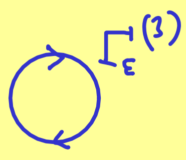


$$\left| \int_{\gamma^{(1)}_{\varepsilon,\delta}} g(z)\,dz - \int_{\Gamma^{(1)}} g(z)\,dz \right| = \left| \int_{(} g(z)\,dz \right| \le \max_{z\in (} |g(z)| \cdot \text{length}(\,(\,) \xrightarrow{\delta\to 0} 0$$

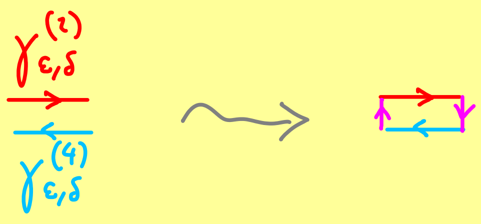
(3)



$\overset{\delta\to 0}{\rightsquigarrow}$



(2) (4)



$$\overset{\text{Cauchy's theorem}}{\Longrightarrow} \oint g(z)\,dz = 0$$

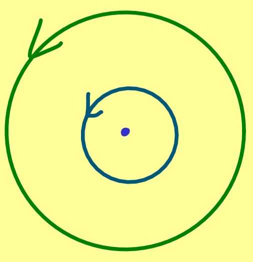
$$\overset{\delta\to 0}{\Longrightarrow} \int_{\rightarrow} g(z)\,dz + \int_{\leftarrow} g(z)\,dz = 0$$

In summary: For $\delta \to 0$:

$$\int_{\Gamma^{(1)}} g(z)\,dz + \int_{\Gamma^{(3)}_{\varepsilon}} g(z)\,dz = 0$$

Result:

$$\int_{\Gamma^{(1)}} g(z)\,dz = \int_{\Gamma^{(3)-}_{\varepsilon}} g(z)\,dz$$

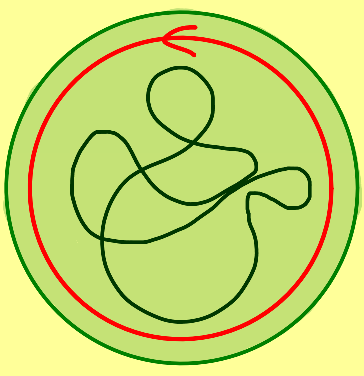


same integral value

The Bright Side of Mathematics

Complex Analysis – Part 27

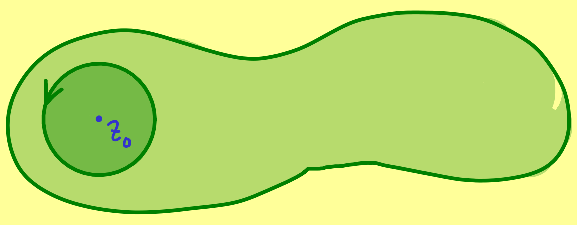
Cauchy's integral formula



$$\oint_\gamma f(z)\,dz = 0$$

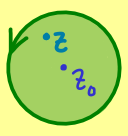
Theorem: $f: D \longrightarrow \mathbb{C}$ holomorphic, $\overline{B_r(z_0)} \subseteq D$,

$\gamma: [a,b] \longrightarrow \mathbb{C}$ closed curve given by the circle on $\partial B_r(z_0)$ $\quad(\text{wind}(\gamma, z_0) = 1)$



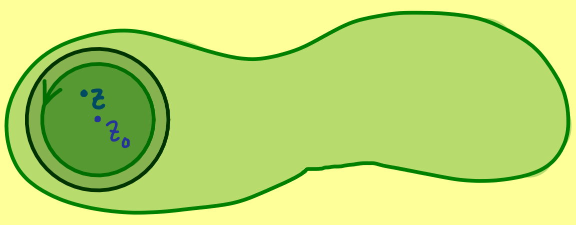
Then:

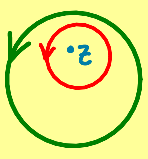
$$\boxed{\begin{array}{l} f(z) = \dfrac{1}{2\pi i} \displaystyle\oint_{\partial B_r(z_0)} \dfrac{f(\zeta)}{\zeta - z}\, d\zeta \\[2ex] \text{for all } z \in B_r(z_0) \end{array}}$$



Proof: $g: B_{\tilde{r}}(z_0) \setminus \{z\} \longrightarrow \mathbb{C}$ holomorphic $\quad \tilde{r} > r$ with $B_{\tilde{r}}(z_0) \subseteq D$

$$\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$$





$$\oint_{\partial B_r(z_0)} g(\zeta)\, d\zeta \overset{\text{last video}}{\underset{\text{keyhole contour}}{=}} \oint_{\partial B_\varepsilon(z)} g(\zeta)\, d\zeta \qquad \text{for all } \varepsilon > 0,\ \varepsilon < r$$

and small enough

$$= \oint_{\partial B_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \oint_{\partial B_\varepsilon(z)} \frac{f(\zeta) - f(z) + f(z)}{\zeta - z}\, d\zeta$$

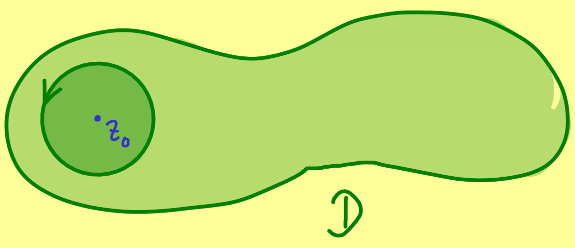
$$= \underbrace{\oint_{\partial B_\varepsilon(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta}_{\left| \oint_{\partial B_\varepsilon(z)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq \max\limits_{\zeta \in \partial B_\varepsilon(z)} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \cdot 2\pi \cdot \varepsilon \xrightarrow{\varepsilon \to 0} 0} + \underbrace{\oint_{\partial B_\varepsilon(z)} \frac{f(z)}{\zeta - z}\, d\zeta}_{2\pi i\, f(z)}$$

$\square$

Complex Analysis – Part 28

Fact: $f: D \to \mathbb{C}$ holomorphic. Then:

(a) $f^{(n)}(z)$ exists for all $z \in D$, $n \in \mathbb{N}$



(b) $$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

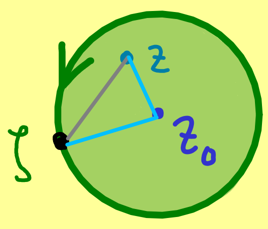
for all $z \in B_r(z_0)$.

(c) In $B_r(z_0)$, f is a power series:

$$f(z) = \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k \quad \text{for} \quad a_k = \frac{1}{k!} \cdot f^{(k)}(z_0)$$

Proof: $$2\pi i \cdot f(z) = \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Cauchy's integral formula



$$= \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} \, d\zeta$$

$$= \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} \cdot \frac{1}{1 - \underbrace{\frac{z - z_0}{\zeta - z_0}}_{=:q}} \, d\zeta \quad , \quad |q| = \frac{\overbrace{|z - z_0|}^{< r}}{\underbrace{|\zeta - z_0|}_{= r}} < 1$$

geometric series

$$= \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} \cdot \sum_{k=0}^{\infty} q^k \, d\zeta$$

uniform convergence

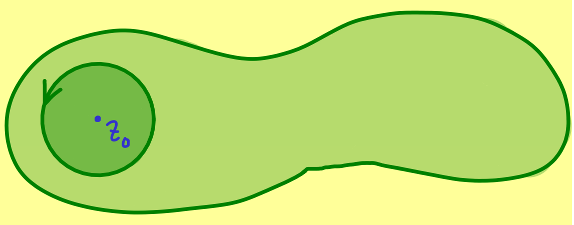
$$= \sum_{k=0}^{\infty} \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} \cdot \left(\frac{z - z_0}{\zeta - z_0} \right)^k d\zeta$$

$$= \sum_{k=0}^{\infty} \tilde{a}_k \cdot (z - z_0)^k \quad \text{for} \quad \tilde{a}_k = \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta$$

Complex Analysis – Part 29

Cauchy's inequalities: $f: D \longrightarrow \mathbb{C}$ holomorphic, $\overline{B_r(z_0)} \subseteq D$.

Then: $\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \cdot \sup_{z \in \partial B_r(z_0)} |f(z)|$



Proof:

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right|$$

parametrized curve: $r \cdot e^{it} + z_0$, $t \in [0, 2\pi]$

$$= \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(r \cdot e^{it} + z_0)}{(r \cdot e^{it})^{n+1}} \cdot r i e^{it} \, dt \right|$$

$$= \left| \frac{n!}{2\pi} \cdot \frac{1}{r^n} \int_0^{2\pi} f(r \cdot e^{it} + z_0) \, e^{it(-n)} \, dt \right|$$

$$\leq \frac{n!}{2\pi} \cdot \frac{1}{r^n} \int_0^{2\pi} \underbrace{\left| f(r \cdot e^{it} + z_0) \right|}_{\leq \sup_{z \in \partial B_r(z_0)} |f(z)|} dt \leq \frac{n!}{\cancel{2\pi}} \cdot \frac{1}{r^n} \cdot \cancel{2\pi} \cdot \sup_{z \in \partial B_r(z_0)} |f(z)|$$

$\square$

Application: $f: \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic and bounded $\left(\sup_{z \in \mathbb{C}} |f(z)| = c \right)$

(Liouville's theorem)

$\Longrightarrow \left| f'(z_0) \right| \leq \frac{1!}{r^1} c$ for all $r > 0$, $z_0 \in \mathbb{C}$

$\Longrightarrow f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$

$\Longrightarrow f: \mathbb{C} \longrightarrow \mathbb{C}$ is constant $\left(\begin{array}{c} \sin: \mathbb{C} \longrightarrow \mathbb{C} \\ \text{not bounded} \end{array} \right)$

Complex Analysis – Part 30

$f: \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic

with known values $\{(z, f(z)) \mid z \in \partial B_r(z_0)\}$

$g: \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic with same values on $\partial B_r(z_0)$

$\overset{\text{Cauchy's integral formula}}{\Longrightarrow}$ $f = g$ on $B_r(z_0)$

Identity theorem: $f, g: \mathbb{D} \longrightarrow \mathbb{C}$ holomorphic, $\mathbb{D} \subseteq \mathbb{C}$ open domain (connected).

Then: $\{z \in \mathbb{D} \mid f(z) = g(z)\}$ has an accumulation point in $\mathbb{D}$

$$\Longleftrightarrow$$

$$f = g$$

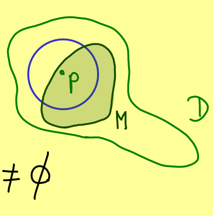
$$\Longleftrightarrow$$

There is $c \in \mathbb{D}$ with $f^{(n)}(c) = g^{(n)}(c)$ for all $n = 0, 1, 2, \dots$

What is an accumulation point?

$p \in \mathbb{D}$ is called an accumulation point of the set $M \subseteq \mathbb{D}$

if for all open set U with $p \in U$: $U \setminus \{p\} \cap M \neq \emptyset$



$M = \mathbb{N}$ no accumulation point

$M = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ 0 is accumulation point

Proof idea: $h := f - g$ holomorphic. Show the equivalence of:

(1) $M = \{z \in \mathbb{D} \mid h(z) = 0\}$ has an accumulation point in $\mathbb{D}$

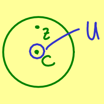
(2) $h = 0$

(3) There is $c \in \mathbb{D}$ with $h^{(n)}(c) = 0$ for all $n = 0, 1, 2, \dots$

(1) $\Rightarrow$ (3) (Contraposition: $\neg(3) \Rightarrow \neg(1)$)

For each $c \in \mathbb{D}$ there is a minimal m with $h^{(m)}(c) \neq 0$

and $h(z) = \sum_{k=m}^{\infty} \underbrace{\frac{h^{(k)}(c)}{k!}}_{a_k} (z-c)^k = \underset{\neq 0}{a_m} \cdot (z-c)^m + \dots$

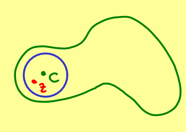


$\Longrightarrow h(z) \neq 0$ for $z \in U \setminus \{c\}$

$\Longrightarrow U \setminus \{c\} \cap M = \emptyset$

(3) $\Rightarrow$ (2) $A_k := \{z \in \mathbb{D} \mid h^{(k)}(z) = 0\}$ closed $\Longrightarrow$ $\underset{\overset{\neq (3)}{\emptyset}}{A} := \bigcap_{k=0}^{\infty} A_k$ closed

A is also open: $c \in A$



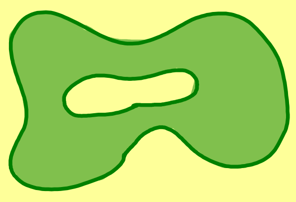
$\overset{\mathbb{D} \text{ connected}}{\Longrightarrow} A = \mathbb{D} \Rightarrow h = 0$

(2) $\Rightarrow$ (1) ✓

□

Complex Analysis – Part 31

Identity theorem: $\mathcal{D} \subseteq \mathbb{C}$ open domain (connected).



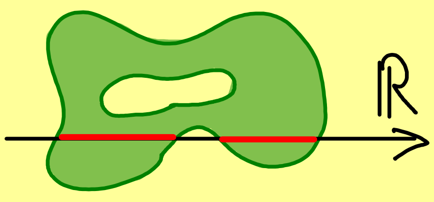
$f, g : \mathcal{D} \longrightarrow \mathbb{C}$ holomorphic.

$$\{ z \in \mathcal{D} \mid f(z) = g(z) \} \text{ has an accumulation point in } \mathcal{D} \quad \Longrightarrow \quad f = g$$

Example: $\cos : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$

Consider a holomorphic function $g : \mathcal{D} \longrightarrow \mathbb{C}$ with $\mathcal{D} \cap \mathbb{R} \neq \emptyset$

and with $g|_{\mathcal{D} \cap \mathbb{R}} = \cos|_{\mathcal{D} \cap \mathbb{R}}$

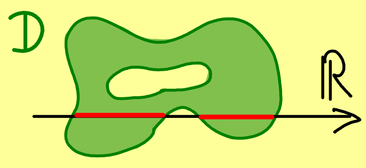


$$\overset{\text{identity theorem}}{\Longrightarrow} \quad g(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \quad \text{for every } z \in \mathcal{D}$$

$\Longrightarrow$ cos has a unique extension for $\mathbb{C}$ as a holomorphic function.

General formulation: $f \in C^{\infty}(\mathbb{R})$ and $\mathcal{D} \subseteq \mathbb{C}$ open domain (connected)

with $\mathcal{D} \cap \mathbb{R} \neq \emptyset$



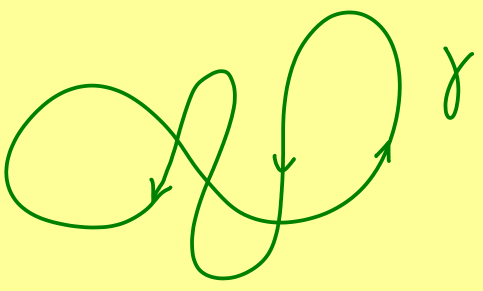
$\Longrightarrow$ there is at most one holomorphic function $g : \mathcal{D} \longrightarrow \mathbb{C}$

with $g|_{\mathcal{D} \cap \mathbb{R}} = f|_{\mathcal{D} \cap \mathbb{R}}$

Complex Analysis – Part 32

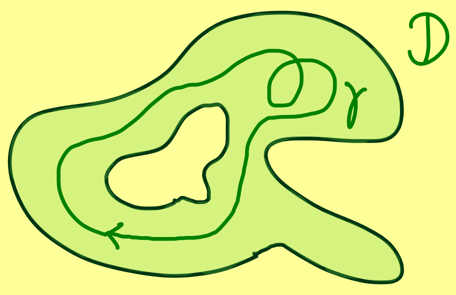
Residue $\rightsquigarrow$ Residue Theorem

Short recapitulation: Closed curve integrals:



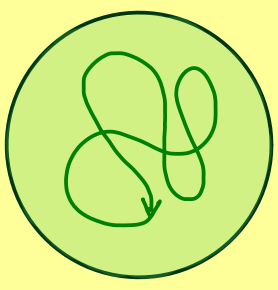
$f: D \to \mathbb{C}$ holomorphic.

(1) $F: D \to \mathbb{C}$ antiderivative of f $\quad (F' = f)$



$$\Longrightarrow \oint_\gamma f(z)\, dz = 0$$

(2) D star domain or $D =$ (annulus)



$$\Longrightarrow \oint_\gamma f(z)\, dz = 0$$

(3) $D = \mathbb{C} \setminus \{z_0\}$, $f(z) = \frac{1}{z - z_0} \Longrightarrow \oint_\gamma f(z)\, dz = 2\pi i \cdot \mathrm{wind}(\gamma, z_0)$

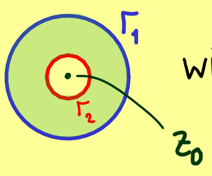
Combine (1) and (3) for Laurent series:

$D =$ (annulus) $= \{ z \in \mathbb{C} \mid r_2 < |z - z_0| < r_1 \}$, $\quad f(z) = \sum_{k=-\infty}^{\infty} a_k \cdot (z - z_0)^k$

$$\Longrightarrow \oint_\gamma f(z)\, dz = a_{-1} \oint_\gamma (z - z_0)^{-1}\, dz = \underbrace{a_{-1}}_{\mathrm{Res}(f, z_0)} \cdot 2\pi i \cdot \mathrm{wind}(\gamma, z_0)$$

residue

Fact: Let f be a Laurent series defined on

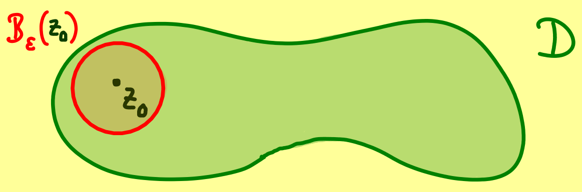


with $r_2 < r < r_1$.

Then: $\mathrm{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} f(z)\, dz$

Definition: Let $f: D \to \mathbb{C}$ be holomorphic and z_0 be an isolated singularity of f.

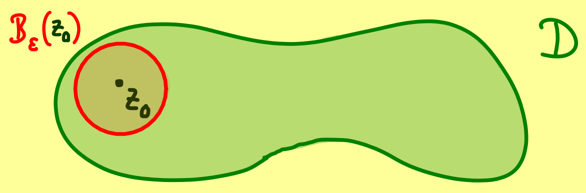
If $B_\varepsilon(z_0) \setminus \{z_0\} \subseteq D$, then we define:



$$\mathrm{Res}(f, z_0) := \frac{1}{2\pi i} \oint_{\partial B_\varepsilon(z_0)} f(z)\, dz$$

residue of f at z_0

Complex Analysis – Part 33



Residue: $\mathrm{Res}(f, z_0) := \frac{1}{2\pi i} \oint_{\partial B_\varepsilon(z_0)} f(z)\, dz$

$\overline{B_\varepsilon(z_0)} \setminus \{z_0\} \subseteq D$

Example: $f: \mathbb{C} \to \mathbb{C}$ holomorphic $\rightsquigarrow$ $\tilde{f}: \mathbb{C}\setminus\{z_0\} \to \mathbb{C}$

$\tilde{f}(z) := f(z)$

$$\mathrm{Res}(f, z_0) := \mathrm{Res}(\tilde{f}, z_0) = \frac{1}{2\pi i} \oint_{\partial B_\varepsilon(z_0)} f(z)\, dz = 0$$

Proposition: $f: D \to \mathbb{C}$ holomorphic, z_0 isolated singularity.

If $f|_{\overline{B_\varepsilon(z_0)} \setminus \{z_0\}}$ is bounded, then $\mathrm{Res}(f, z_0) = 0$.

Proof: $$\left| \oint_{\partial B_\varepsilon(z_0)} f(z)\, dz \right| \leq \underbrace{\max_{z \in \mathrm{Ran}(\gamma)} |f(z)|}_{\leq C} \cdot \underbrace{\mathrm{length}(\gamma)}_{2\pi\varepsilon} \xrightarrow{\varepsilon \to 0} 0$$

$$\Longrightarrow \mathrm{Res}(f, z_0) = 0$$

Residue for poles

$f: D \to \mathbb{C}$ holomorphic, z_0 isolated singularity.

z_0 pole $:\Longleftrightarrow$ the function $h: B_\varepsilon(z_0) \to \mathbb{C}$ with $h(z) = \frac{1}{f(z)}$, $h(z_0) = 0$ is holomorphic

Example: $f(z) = \frac{1}{z - z_0} \rightsquigarrow h(z) = z - z_0$

pole $\leftarrow$ holomorphic

Fact: $f: D \to \mathbb{C}$ has a pole at z_0 (of order N)

$\Longleftrightarrow$ There is a unique $N \in \mathbb{N}$ and non-vanishing holomorphic function $g: B_\varepsilon(z_0) \to \mathbb{C}$ such that

$$f(z) = (z - z_0)^{-N} \cdot g(z) \quad \text{for } z \in B_\varepsilon(z_0)$$

$\Longleftrightarrow$ There is a unique $N \in \mathbb{N}$ and a holomorphic function $\tilde{g}: B_\varepsilon(z_0) \to \mathbb{C}$:

$$f(z) = \frac{a_{-N}}{(z - z_0)^N} + \cdots + \frac{a_{-1}}{(z - z_0)^1} + \tilde{g}(z) \quad \text{for } z \in B_\varepsilon(z_0)$$

$a_{-N} \neq 0$

Theorem: $f: D \to \mathbb{C}$ holomorphic, z_0 isolated singularity.

If z_0 is a pole of order N, then:

$(N-1)$th complex derivative

$$\mathrm{Res}(f, z_0) = \frac{1}{(N-1)!} \lim_{z \to z_0} \left(\frac{d}{dz}\right)^{N-1} (z - z_0)^N f(z)$$

Example: $f(z) = \frac{1}{z^2(1+z)}$, $z_0 = 0$ is a pole order 2

$$\mathrm{Res}(f, z_0) = \frac{1}{1!} \lim_{z \to 0} \left(\frac{d}{dz}\right)(z - 0)^2 f(z) = \lim_{z \to 0} \frac{d}{dz}\left(\frac{1}{1+z}\right)$$

$$= \lim_{z \to 0} \left(-\frac{1}{(1+z)^2}\right) = -1$$

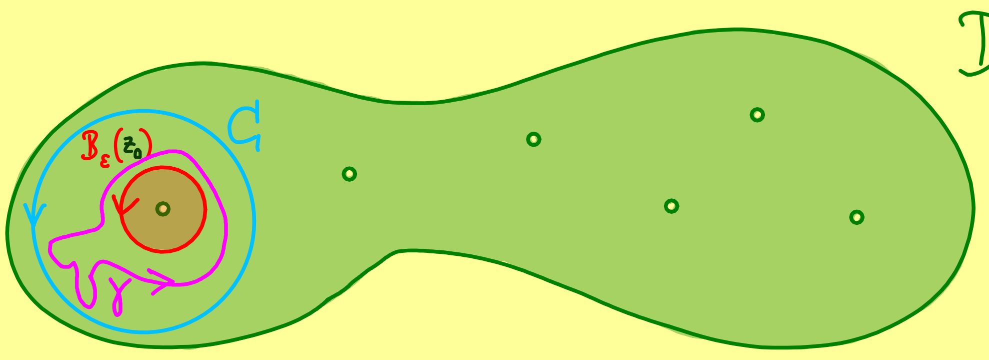
The Bright Side of Mathematics

Complex Analysis – Part 34

$f: D \to \mathbb{C}$

holomorphic

isolated singularity z_0

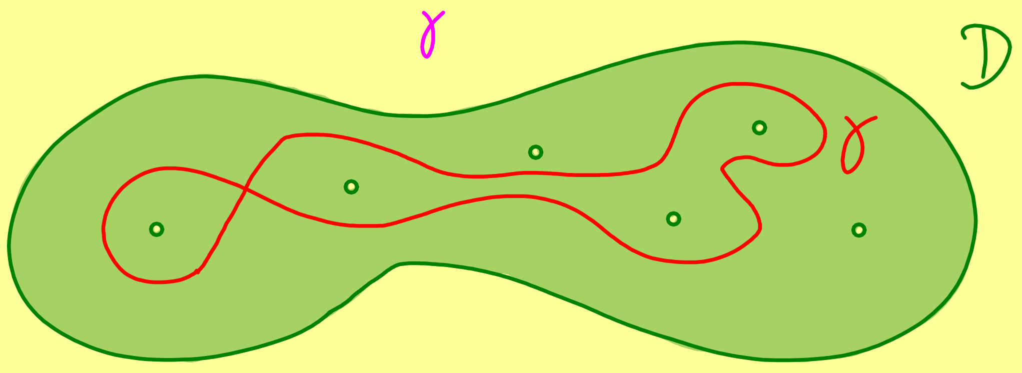


$$\mathrm{Res}(f, z_0) := \frac{1}{2\pi i} \oint_{\partial B_\varepsilon(z_0)} f(z)\, dz$$

$$\Rightarrow \quad 2\pi i \cdot \mathrm{Res}(f, z_0) = \oint_{\partial B_\varepsilon(z_0)} f(z)\, dz = \oint_{C} f(z)\, dz$$

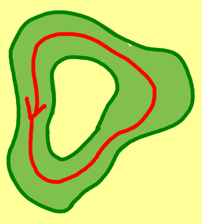
circle around z_0

$$\Rightarrow \quad \mathrm{wind}(\gamma, z_0) \cdot 2\pi i \cdot \mathrm{Res}(f, z_0) = \oint_{\gamma} f(z)\, dz$$



Residue theorem: $D \subseteq \mathbb{C}$ open domain, (connected) $f: D \to \mathbb{C}$ holomorphic, $z_1, z_2, \dots, z_n$ isolated singularities of f, $\gamma: [a,b] \to D$ closed curve with $\mathrm{Int}(\gamma) \subseteq D \cup \{z_1, z_2, \dots, z_n\}$.

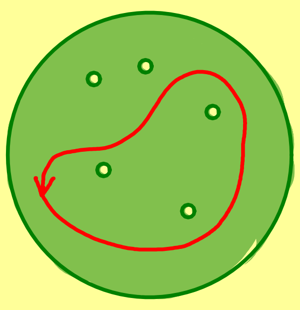
not allowed:

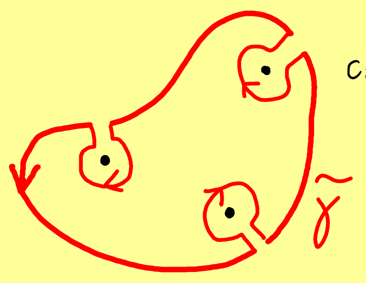


Then:

$$\oint_{\gamma} f(z)\, dz = \sum_{j=1}^{n} 2\pi i \cdot \mathrm{wind}(\gamma, z_j) \cdot \mathrm{Res}(f, z_j)$$

Proof: $\tilde{D} \subseteq \mathbb{C}$ open disc, $D = \tilde{D} \setminus \{z_1, z_2, \dots, z_n\}$.





Cauchy's theorem $\rightsquigarrow$ $\oint_{\tilde{\gamma}} f(z)\, dz = 0$

The Bright Side of Mathematics

Complex Analysis – Part 35

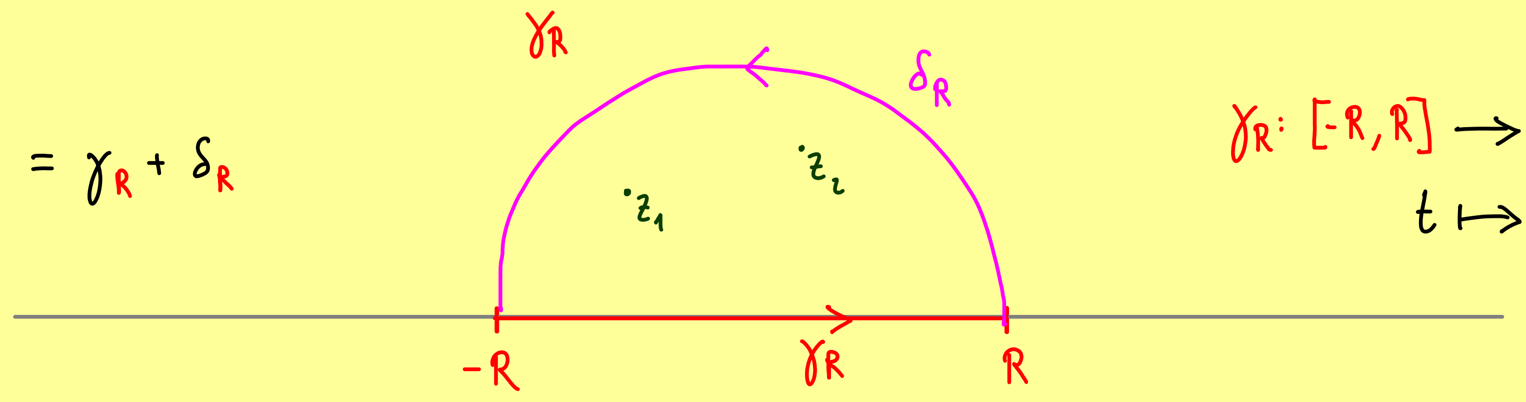
$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx = ?$$

$$= \lim_{R\to\infty} \underbrace{\int_{-R}^{-R} f(x)\,dx}_{=\int_{\gamma_R} f(z)\,dz} \qquad \text{where } f(x) = \frac{x^4}{1+x^6}$$

complex contour integral:

$$\int_{\gamma_R} f(z)\,dz \qquad \text{where } f(z) = \frac{z^4}{1+z^6}$$

$$\Gamma_R = \gamma_R + \delta_R$$



$$\gamma_R : [-R, R] \to \mathbb{C}$$
$$t \mapsto t$$

residue theorem:

$$\oint_{\Gamma_R} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)$$

$$\| $$

$$\int_{\gamma_R} f(z)\,dz + \int_{\delta_R} f(z)\,dz \qquad \text{where } \left| \int_{\delta_R} f(z)\,dz \right| \leq \max_{z\in \operatorname{Ran}(\delta_R)} \underbrace{|f(z)|}_{=\frac{(Re^{it})^4}{1+(Re^{it})^6}} \cdot \operatorname{length}(\delta_R)$$

$$\leq C \cdot \frac{1}{R^{\cancel{2}}} \cdot \pi \cdot \cancel{R} \xrightarrow{R\to\infty} 0$$

Hence:

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty} \oint_{\Gamma_R} f(z)\,dz$$

$$= 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(f, z)$$

poles: $1+x^6 = 0 \;\Rightarrow\; z_1 = e^{i\frac{\pi}{6}},\; z_2 = e^{3i\frac{\pi}{6}},\; z_3 = e^{5i\frac{\pi}{6}}$

formula for simple poles: $\operatorname{Res}\left(\frac{h}{g}, z\right) = \frac{h(z)}{g'(z)}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx = 2\pi i \cdot \left(\frac{1}{6} e^{-i\frac{\pi}{6}} + \frac{1}{6} e^{-3i\frac{\pi}{6}} + \frac{1}{6} e^{-5i\frac{\pi}{6}} \right)$$

$$= \frac{1}{3}\pi i \left(i \underbrace{\sin\left(-\frac{\pi}{6}\right)}_{-\sin\left(\frac{\pi}{6}\right)} + i \underbrace{\sin\left(-\frac{3\pi}{6}\right)}_{=-1} + i \underbrace{\sin\left(-\frac{5\pi}{6}\right)}_{-\sin\left(\frac{\pi}{6}\right)} \right)$$

$$= \frac{1}{3}\pi \left(1 + 2\sin\left(\frac{\pi}{6}\right) \right)$$