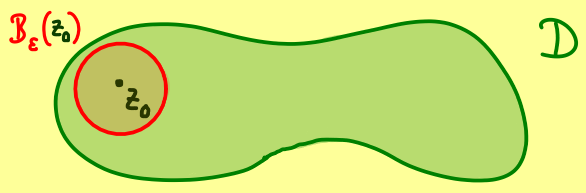
BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Complex Analysis – Part 33



Residue: $\mathrm{Res}(f,z_0) := \frac{1}{2\pi i}\oint_{\partial B_\varepsilon(z_0)} f(z)\,dz$

$\overline{B_\varepsilon(z_0)}\setminus\{z_0\} \subseteq D$

Example: $f:\mathbb{C}\to\mathbb{C}$ holomorphic $\rightsquigarrow$ $\tilde{f}:\mathbb{C}\setminus\{z_0\}\to\mathbb{C}$

$\tilde{f}(z) := f(z)$

$$\mathrm{Res}(f,z_0) := \mathrm{Res}(\tilde{f},z_0) = \frac{1}{2\pi i}\oint_{\partial B_\varepsilon(z_0)} f(z)\,dz = 0$$

Proposition: $f: D \to \mathbb{C}$ holomorphic, z_0 isolated singularity.

If $f|_{\overline{B_\varepsilon(z_0)}\setminus\{z_0\}}$ is bounded, then $\mathrm{Res}(f,z_0) = 0$.

Proof: $$\left|\oint_{\partial B_\varepsilon(z_0)} f(z)\,dz\right| \leq \underbrace{\max_{z\in \mathrm{Ran}(\gamma)} |f(z)|}_{\leq C} \cdot \underbrace{\mathrm{length}(\gamma)}_{2\pi\varepsilon} \xrightarrow{\varepsilon\to 0} 0$$

$$\Longrightarrow \mathrm{Res}(f,z_0) = 0$$

Residue for poles

$f: D\to\mathbb{C}$ holomorphic, z_0 isolated singularity.

z_0 pole $:\Longleftrightarrow$ the function $h: B_\varepsilon(z_0)\to\mathbb{C}$ with $h(z) = \frac{1}{f(z)}$, $h(z_0) = 0$

is holomorphic

Example: $f(z) = \frac{1}{z-z_0} \rightsquigarrow h(z) = z - z_0$

pole $\Leftarrow$ holomorphic

Fact: $f: D\to\mathbb{C}$ has a pole at z_0 (of order N)

$\Longleftrightarrow$ There is a unique $N\in\mathbb{N}$ and non-vanishing holomorphic function $g: B_\varepsilon(z_0)\to\mathbb{C}$ such that

$$f(z) = (z-z_0)^{-N}\cdot g(z) \quad \text{for } z\in B_\varepsilon(z_0)$$

$\Longleftrightarrow$ There is a unique $N\in\mathbb{N}$ and a holomorphic function $\tilde{g}: B_\varepsilon(z_0)\to\mathbb{C}$:

$$f(z) = \frac{\overset{\neq 0}{a_{-N}}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{(z-z_0)^1} + \tilde{g}(z) \quad \text{for } z\in B_\varepsilon(z_0)$$

Theorem: $f: D\to\mathbb{C}$ holomorphic, z_0 isolated singularity.

If z_0 is a pole of order N, then:

(N-1)th complex derivative

$$\mathrm{Res}(f,z_0) = \frac{1}{(N-1)!}\lim_{z\to z_0}\left(\frac{d}{dz}\right)^{N-1}(z-z_0)^N f(z)$$

Example: $f(z) = \frac{1}{z^2(1+z)}$, $z_0 = 0$ is a pole order 2

$$\mathrm{Res}(f,z_0) = \frac{1}{1!}\lim_{z\to 0}\left(\frac{d}{dz}\right)(z-0)^2 f(z) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{1+z}\right)$$

$$= \lim_{z\to 0}\left(-\frac{1}{(1+z)^2}\right) = -1$$