

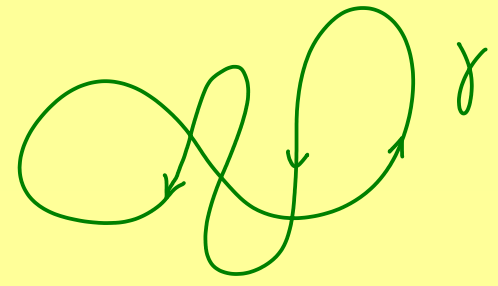


Complex Analysis - Part 32

Residue \rightsquigarrow Residue Theorem

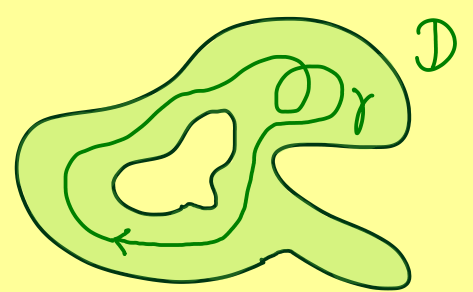
Short recapitulation: Closed curve integrals:

$f: \mathcal{D} \rightarrow \mathbb{C}$ holomorphic.



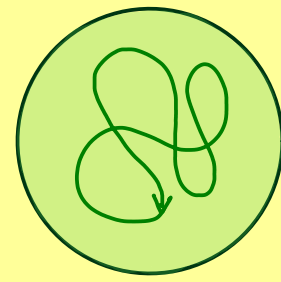
(1) $F: \mathcal{D} \rightarrow \mathbb{C}$ antiderivative of f ($F' = f$)

$$\Rightarrow \oint_{\gamma} f(z) dz = 0$$



(2) \mathcal{D} star domain or $\mathcal{D} = \text{annulus}$

$$\Rightarrow \oint_{\gamma} f(z) dz = 0$$



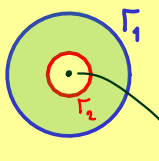
(3) $\mathcal{D} = \mathbb{C} \setminus \{z_0\}$, $f(z) = \frac{1}{z-z_0} \Rightarrow \oint_{\gamma} f(z) dz = 2\pi i \cdot \text{wind}(\gamma, z_0)$

Combine (1) and (3) for Laurent series:

$$\mathcal{D} = \text{annulus} = \{z \in \mathbb{C} \mid r_2 < |z-z_0| < r_1\}, \quad f(z) = \sum_{k=-\infty}^{\infty} a_k \cdot (z-z_0)^k$$

$$\Rightarrow \oint_{\gamma} f(z) dz = a_{-1} \oint_{\gamma} (z-z_0)^{-1} dz = a_{-1} \cdot 2\pi i \cdot \text{wind}(\gamma, z_0)$$

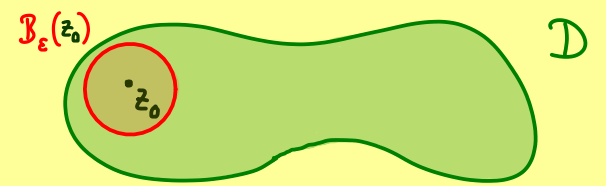
$\underbrace{\hspace{10em}}_{\text{Res}(f, z_0) \text{ residue}}$

Fact: Let f be a Laurent series defined on  with $r_2 < r < r_1$.

Then: $\text{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \oint_{\partial \mathcal{B}_r(z_0)} f(z) dz$

Definition: Let $f: \mathcal{D} \rightarrow \mathbb{C}$ be holomorphic and z_0 be an isolated singularity of f .

If $\overline{\mathcal{B}_\varepsilon(z_0)} \setminus \{z_0\} \subseteq \mathcal{D}$, then we define:



$$\text{Res}(f, z_0) := \frac{1}{2\pi i} \oint_{\partial \mathcal{B}_\varepsilon(z_0)} f(z) dz$$

residue of f at z_0