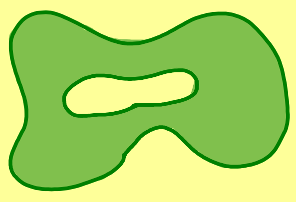
The Bright Side of Mathematics

Complex Analysis – Part 31

Identity theorem: $\mathcal{D} \subseteq \mathbb{C}$ open domain (connected).



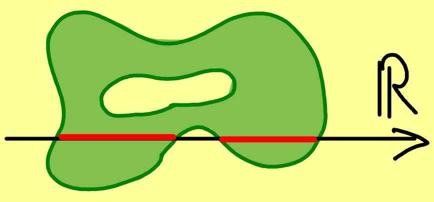
$f, g : \mathcal{D} \longrightarrow \mathbb{C}$ holomorphic.

$\{ z \in \mathcal{D} \mid f(z) = g(z) \}$ has an accumulation point in $\mathcal{D}$ $\Longrightarrow$ $f = g$

Example: $\cos : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$

Consider a holomorphic function $g : \mathcal{D} \longrightarrow \mathbb{C}$ with $\mathcal{D} \cap \mathbb{R} \neq \emptyset$

and with $g|_{\mathcal{D} \cap \mathbb{R}} = \cos|_{\mathcal{D} \cap \mathbb{R}}$

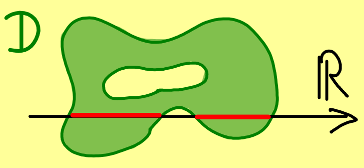


$\overset{\text{identity theorem}}{\Longrightarrow}$ $g(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ for every $z \in \mathcal{D}$

$\Longrightarrow$ cos has a unique extension for $\mathbb{C}$ as a holomorphic function.

General formulation: $f \in C^{\infty}(\mathbb{R})$ and $\mathcal{D} \subseteq \mathbb{C}$ open domain (connected)

with $\mathcal{D} \cap \mathbb{R} \neq \emptyset$



$\Longrightarrow$ there is at most one holomorphic function $g : \mathcal{D} \longrightarrow \mathbb{C}$

with $g|_{\mathcal{D} \cap \mathbb{R}} = f|_{\mathcal{D} \cap \mathbb{R}}$