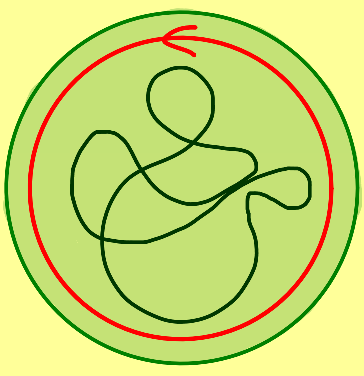
The Bright Side of Mathematics

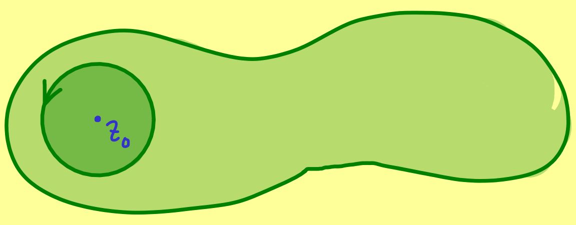
Complex Analysis – Part 27

Cauchy's integral formula



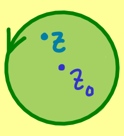
$$\oint_{\gamma} f(z)\,dz = 0$$

Theorem: $f: D \longrightarrow \mathbb{C}$ holomorphic, $\overline{B_r(z_0)} \subseteq D$,



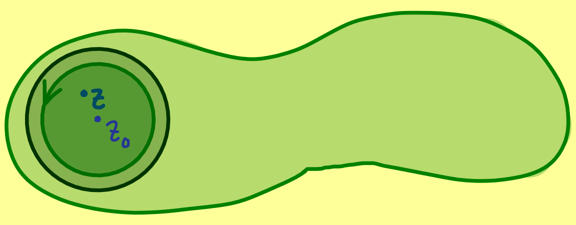
$\gamma: [a,b] \longrightarrow \mathbb{C}$ closed curve given by the circle on $\partial B_r(z_0)$ $\quad(\text{wind}(\gamma, z_0) = 1)$

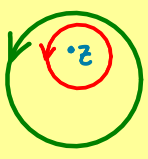
Then:
$$\boxed{f(z) = \frac{1}{2\pi i} \oint_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad \text{for all } z \in B_r(z_0)}$$



Proof: $g: B_{\tilde r}(z)\setminus\{z\} \longrightarrow \mathbb{C}$ holomorphic $\quad \tilde r > r$

$$\zeta \longmapsto \frac{f(\zeta)}{\zeta - z} \quad \text{with } B_{\tilde r}(z_0) \subseteq D$$





$$\oint_{\partial B_r(z_0)} g(\zeta)\,d\zeta \overset{\text{last video}}{\underset{\text{keyhole contour}}{=}} \oint_{\partial B_\varepsilon(z)} g(\zeta)\,d\zeta \quad \text{for all } \varepsilon > 0,\ \varepsilon < r$$

$$= \oint_{\partial B_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \oint_{\partial B_\varepsilon(z)} \frac{f(\zeta) - f(z) + f(z)}{\zeta - z}\,d\zeta$$

$$= \underbrace{\oint_{\partial B_\varepsilon(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta}_{} + \underbrace{\oint_{\partial B_\varepsilon(z)} \frac{f(z)}{\zeta - z}\,d\zeta}_{2\pi i\, f(z)}$$

$$\left| \oint_{\partial B_\varepsilon(z)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta \right| \leq \max_{\zeta \in \partial B_\varepsilon(z)} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \cdot 2\pi \cdot \varepsilon \xrightarrow{\varepsilon \to 0} 0$$

$\square$