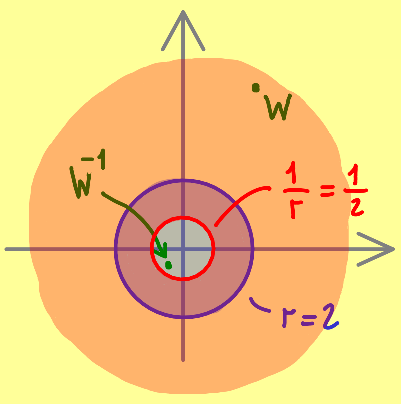
The Bright Side of Mathematics

Complex Analysis – Part 15

Laurent series (generalisation of power series + holomorphic)

$\sum_{k=0}^{\infty} a_k \cdot z^k$ with radius of convergence $r \in [0,\infty]$

$\sum_{k=0}^{\infty} a_k \cdot \left(\frac{1}{w}\right)^k$ is convergent $\left\{ \begin{array}{l} \left|\frac{1}{w}\right| < r \\ \Leftrightarrow \\ |w| > \frac{1}{r} \end{array} \right.$

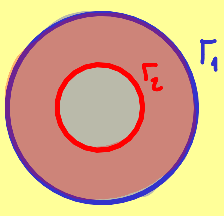


chain rule

$\Rightarrow \quad w \mapsto \sum_{k=0}^{\infty} a_k \cdot w^{-k}$ is holomorphic on $\mathbb{C} \setminus \overline{B_{\frac{1}{r}}(0)}$

(alternatively: constant + $\sum_{k=-1}^{-\infty} b_k \cdot z^k$)

Combine two series:



$z \mapsto \sum_{k=0}^{\infty} a_k \cdot z^k \rightsquigarrow$ with radius of convergence r_1

$\sum_{k=1}^{\infty} b_{-k} \cdot z^k \rightsquigarrow$ with radius of convergence r

$z \mapsto \sum_{k=-1}^{-\infty} b_k \cdot z^k \rightsquigarrow$ with "radius of convergence" $r_2 = \frac{1}{r}$

Definition: A Laurent series written as $\sum_{k=-\infty}^{\infty} a_k \cdot (z - z_0)^k$ is a pair of two series:

$z \mapsto \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^k$ with radius of convergence $r_1 \in [0,\infty]$

principal part $\hookrightarrow$ $z \mapsto \sum_{k=-1}^{-\infty} a_k \cdot (z - z_0)^k$ with "radius of convergence" $r_2 \in [0,\infty]$

a_{-1} is called the residue of the Laurent series.

The Laurent series is a holomorphic function on $\{ z \in \mathbb{C} \mid r_2 < |z - z_0| < r_1 \}$