Calculating the kernel of matrix

(an example)

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 4 \\ 4 & -4 & 0 & 8 \end{pmatrix} \in \mathbb{R}^{3\times 4}$$

$$\ker(A) := \{ x \in \mathbb{R}^4 \mid Ax = 0 \}$$

$$\ker(A) = \ker \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 4 \\ 4 & -4 & 0 & 8 \end{pmatrix} \begin{matrix} \\ \leftarrow \text{II} - 3\,\text{I} \\ \leftarrow \text{III} - 4\,\text{I} \end{matrix}$$

$$= \ker \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & -8 & -8 & 4 \end{pmatrix} \leftarrow \text{III} - 4\,\text{II}$$

$$= \ker \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} -\alpha - \frac{3}{2}\beta \\ -\alpha + \frac{1}{2}\beta \\ \alpha \\ \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \alpha \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

basis of this $\ker(A)$

$-2x_2 - 2x_3 + x_4 = 0$ ($x_3 = \alpha$, $x_4 = \beta$)

$\Rightarrow x_2 = -\alpha + \frac{1}{2}\beta$

$x_1 + x_2 + 2x_3 + x_4 = 0$

$x_2 = -\alpha + \frac{1}{2}\beta$, $x_3 = \alpha$, $x_4 = \beta$

$x_1 = -\alpha - \frac{3}{2}\beta$