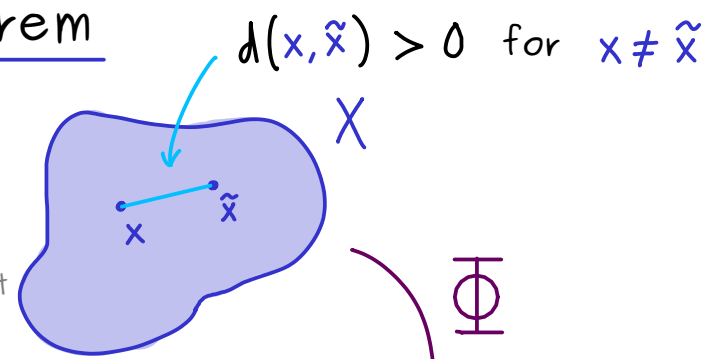


## Banach fixed-point theorem

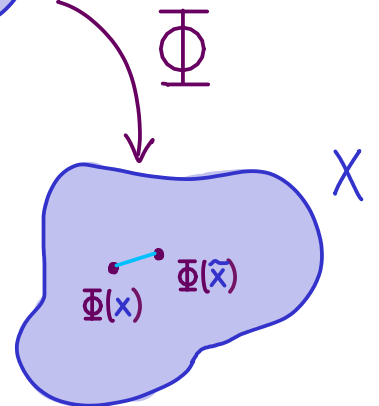
Let  $(X, d)$  be a complete metric space

metric = distance function  
all Cauchy sequences are convergent



and  $\Phi : X \rightarrow X$  be a contraction, which means:

$$\exists q \in [0, 1) \quad \forall x, \tilde{x} \in X : d(\Phi(x), \Phi(\tilde{x})) \leq q \cdot d(x, \tilde{x})$$

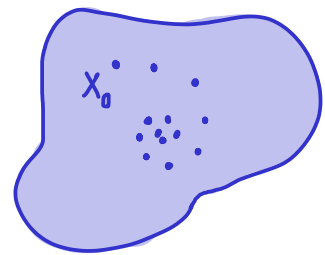


Then:  $\Phi$  has a unique fixed point  $x^* \in X$  ( $\Phi(x^*) = x^*$ )

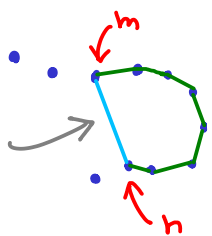
and for each  $x_0 \in X$  we have:  $\Phi^n(x_0) \xrightarrow{n \rightarrow \infty} x^*$ .

Proof: For a given  $x_0 \in X$ , define  $x_n := \Phi^n(x_0)$

Is  $(x_n)_{n \in \mathbb{N}}$  a Cauchy sequence?



$$\begin{aligned} d(x_{n+1}, x_n) &= d(\Phi(x_n), \Phi(x_{n-1})) \leq q \cdot d(x_n, x_{n-1}) \\ &= q \cdot d(\Phi(x_{n-1}), \Phi(x_{n-2})) \leq q^2 \cdot d(x_{n-1}, x_{n-2}) \\ &\dots \leq q^n \cdot d(x_1, x_0) \quad (\text{proof by induction}) \end{aligned}$$



$\Delta$ -inequality

$$\text{For } n > m : d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m)$$

$$\leq (q^{n-1} + q^{n-2} + \dots + q^m) \cdot d(x_1, x_0)$$

$$= q^m \cdot \sum_{k=0}^{n-1-m} q^k \cdot d(x_1, x_0)$$

$$\leq \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

$$\leq \frac{q^m}{1-q} \cdot d(x_1, x_0)$$

$\Rightarrow (x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence  $\left( d(x_n, x_m) \xrightarrow{n, m \rightarrow \infty} 0 \right)$

completeness

$\Rightarrow (x_n)_{n \in \mathbb{N}}$  has a unique limit  $x^* \in X$

Fixed point?  $\Phi(x^*) = \Phi(\lim_{n \rightarrow \infty} x_n) \stackrel{\text{contraction is continuous}}{=} \lim_{n \rightarrow \infty} \Phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$

Uniqueness? We have a map  $\Phi$  with  $d(\Phi(x), \Phi(\hat{x})) \leq q \cdot d(x, \hat{x})$  and fixed points  $x^*, \hat{x}$ .

$$x^* \neq \hat{x} \Rightarrow d(x^*, \hat{x}) = d(\Phi(x^*), \Phi(\hat{x})) \leq q \cdot d(x^*, \hat{x})$$

$$\Rightarrow 1 \leq q$$

By contraposition:  $0 \leq q < 1 \Rightarrow x^* = \hat{x} \quad \square$