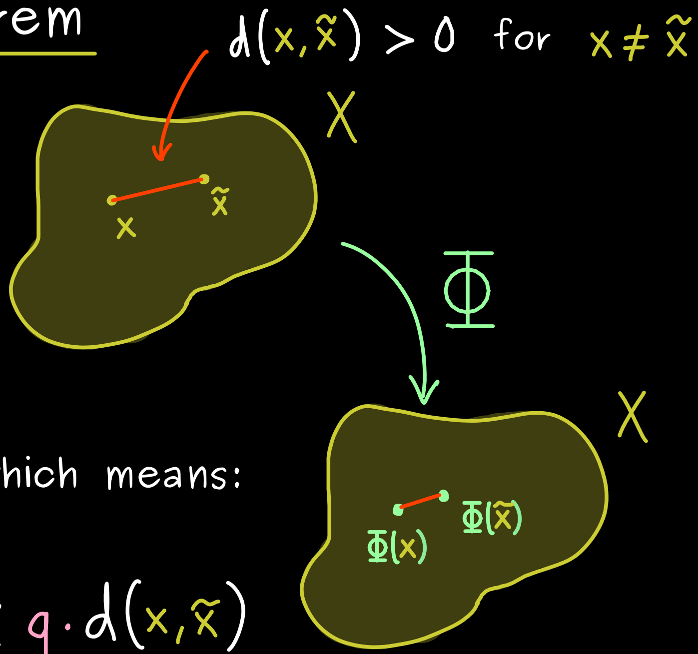
Banach fixed-point theorem



Let (X, d) be a complete metric space

all Cauchy sequences are convergent

metric = distance function

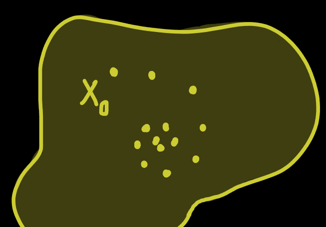
and $\Phi : X \longrightarrow X$ be a contraction, which means:

$$\exists\, q \in [0,1) \;\; \forall x, \tilde{x} \in X : \;\; d\big(\Phi(x), \Phi(\tilde{x})\big) \leq q \cdot d(x, \tilde{x})$$

Then: Φ has a unique fixed point $x^* \in X$ $\quad \big(\Phi(x^*) = x^* \big)$

and for each $x_0 \in X$ we have: $\Phi^n(x_0) \xrightarrow{n \to \infty} x^*$.

Proof: For a given $x_0 \in X$, define $x_n := \Phi^n(x_0)$

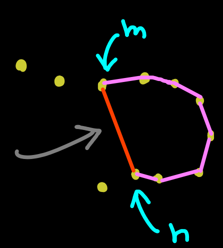


Is $(x_n)_{n \in \mathbb{N}}$ a Cauchy sequence?

$$d(x_{n+1}, x_n) = d\big(\Phi(x_n), \Phi(x_{n-1})\big) \leq q \cdot d(x_n, x_{n-1})$$

$$= q \cdot d\big(\Phi(x_{n-1}), \Phi(x_{n-2})\big) \leq q^2 \cdot d(x_{n-1}, x_{n-2})$$

$$\cdots \leq q^n \cdot d(x_1, x_0) \qquad \text{(proof by induction)}$$



For $n > m$: $d(x_n, x_m) \overset{\Delta\text{-inequality}}{\leq} d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots$

$$\cdots + d(x_{m+1}, x_m)$$

$$\leq \left(q^{n-1} + q^{n-2} + \cdots + q^m \right) \cdot d(x_1, x_0)$$

$$= q^m \cdot \underbrace{\sum_{k=0}^{n-1-m} q^k}_{\leq \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}} \cdot d(x_1, x_0)$$

$$\leq \frac{q^m}{1-q} \cdot d(x_1, x_0)$$

$\Longrightarrow \quad (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence $\quad \left(d(x_n, x_m) \xrightarrow{n,m\to\infty} 0 \right)$

$\overset{\text{completeness}}{\Longrightarrow} \quad (x_n)_{n\in\mathbb{N}}$ has a unique limit $x^* \in X$

Fixed point? $\quad \Phi(x^*) = \Phi\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$

contraction is continuous

Uniqueness? $\quad$ We have a map Φ with $d(\Phi(x), \Phi(\tilde{x})) \leq q \cdot d(x, \tilde{x})$ and fixed points $x^*, \hat{x}$.

$$x^* \neq \hat{x} \Longrightarrow d(x^*, \hat{x}) = d(\Phi(x^*), \Phi(\hat{x})) \leq q \cdot d(x^*, \hat{x})$$

$$\Longrightarrow \quad 1 \leq q$$

By contraposition: $\quad 0 \leq q < 1 \Longrightarrow x^* = \hat{x}$ $\quad \square$