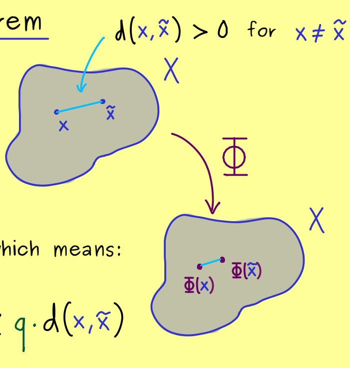
The Bright Side of Mathematics

Banach fixed-point theorem



Let (X, d) be a complete metric space

(complete: all Cauchy sequences are convergent; metric = distance function)

and $\Phi : X \to X$ be a contraction, which means:

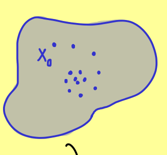
$$\exists q \in [0,1) \; \forall x, \tilde{x} \in X : \; d(\Phi(x), \Phi(\tilde{x})) \leq q \cdot d(x, \tilde{x})$$

Then: Φ has a unique fixed point $x^* \in X$ $\quad (\Phi(x^*) = x^*)$

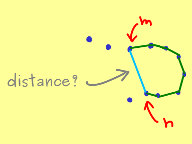
and for each $x_0 \in X$ we have: $\Phi^n(x_0) \xrightarrow{n \to \infty} x^*$.

Proof: For a given $x_0 \in X$, define $x_n := \Phi^n(x_0)$

Is $(x_n)_{n \in \mathbb{N}}$ a Cauchy sequence?



$$d(x_{n+1}, x_n) = d(\Phi(x_n), \Phi(x_{n-1})) \leq q \cdot d(x_n, x_{n-1})$$
$$= q \cdot d(\Phi(x_{n-1}), \Phi(x_{n-2})) \leq q^2 \cdot d(x_{n-1}, x_{n-2})$$
$$\cdots \leq q^n \cdot d(x_1, x_0) \quad \text{(proof by induction)}$$



For $n > m$: (Δ-inequality)

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$
$$\leq (q^{n-1} + q^{n-2} + \cdots + q^m) \cdot d(x_1, x_0)$$
$$= q^m \cdot \sum_{k=0}^{n-1-m} q^k \cdot d(x_1, x_0)$$
$$\sum_{k=0}^{n-1-m} q^k \leq \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$
$$\leq \frac{q^m}{1-q} \cdot d(x_1, x_0)$$

$\Rightarrow$ $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence $\quad \left(d(x_n, x_m) \xrightarrow{n,m \to \infty} 0 \right)$

$\overset{\text{completeness}}{\Rightarrow}$ $(x_n)_{n \in \mathbb{N}}$ has a unique limit $x^* \in X$

Fixed point? $\quad \Phi(x^*) = \Phi(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} \Phi(x_n) = \lim_{n \to \infty} x_{n+1} = x^*$

(contraction is continuous)

Uniqueness? We have a map Φ with $d(\Phi(x), \Phi(\tilde{x})) \leq q \cdot d(x, \tilde{x})$ and fixed points $x^*, \hat{x}$.

$$x^* \neq \hat{x} \Rightarrow d(x^*, \hat{x}) = d(\Phi(x^*), \Phi(\hat{x})) \leq q \cdot d(x^*, \hat{x})$$
$$\Rightarrow 1 \leq q$$

By contraposition: $0 \leq q < 1 \Rightarrow x^* = \hat{x}$ $\quad \square$