BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Baire Category Theorem

(X,d) complete metric space

A countable intersection of dense open sets is dense!

$Q_j \subseteq X$ dense $(\overline{Q_j} = X)$ + open

$\Longrightarrow \bigcap_{j=1}^{\infty} Q_j$ is also dense

Notions: Let $(X,\mathcal{T})$ be a topological space.

(1) $M \subseteq X$ nowhere dense (in X) if $(\overline{M})^\circ = \emptyset$

(2) $M \subseteq X$ meagre (in X) (set of first category (in X)) $:\Longleftrightarrow M = \bigcup_{j=1}^{\infty} S_j$ ← nowhere dense

(3) $M \subseteq X$ nonmeagre (in X) (set of second category (in X)) $:\Longleftrightarrow M$ is not meagre

Baire category theorem: (X,d) complete metric space

$\Longrightarrow$ each open set $(\neq \emptyset)$ is of second category (in X)

Special version: (X,d) complete metric space $\Longrightarrow X$ is of second category (in X)

Application: $C([0,1]) = \bigcup_{j=1}^{\infty} \overbrace{A_j}^{\text{nowhere dense}} \cup B \subseteq$ {functions that are nowhere differentiable}

$\bigcup_{j=1}^{\infty} A_j \supseteq$ {functions that are differentiable at least at one point in $[0,1]$}

$\overset{\text{Baire}}{\Longrightarrow} B \neq \emptyset$ nonmeagre, B is dense in $C([0,1])$