The Bright Side of Mathematics

The following pages cover the whole Algebra course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Algebra – Part 1

Start Learning Mathematics

Real Analysis

Linear Algebra

Algebra

- groups
- rings
- fields

group: $(\mathbb{Z}, +)$

ring: $(\mathbb{Z}, +, \cdot)$

field: $(\mathbb{R}, +, \cdot)$

Galois theory

$$x^5 - 4x + 2 = 0$$

at most 5 solutions

solutions cannot be expressed with combinations of roots

Algebra - Part 2

Definition: Let A be a set.

A map $F: A \times A \longrightarrow A$ is called a binary operation on A.

Instead of $F((a,b))$, we write $a \circ b$ or $a * b$ or $a F b$

or $a \cdot b$ or ab or $a+b$ ...

juxtaposition

Closure Law: $a \circ b \in A$ for all $a, b \in A$

Example: $A = \{1,2,3\}$, $\circ : A \times A \longrightarrow A$ binary operation defined by:

operation table:

$\circ$	1	2	3
1	3	1	2
2	3	3	1
3	2	2	2

$1 \circ 2 = 1$
$2 \circ 1 = 3$
not equal!

$(1 \circ 2) \circ 3 = 1 \circ 3 = 2$
$1 \circ (2 \circ 3) = 1 \circ 1 = 3$
not equal!

Definition: A pair $(S, \circ)$ where S is a set and $\circ$ is a binary operation on S

is called a semigroup if

$$a \circ (b \circ c) = (a \circ b) \circ c \quad \text{for all } a,b,c \in S \qquad \text{(associative)}$$

$$=: a \circ b \circ c$$

<u>Example:</u> Set of functions $\mathcal{F}(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ function} \}$

together with composition $\circ : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R})$:

Take $f_1, f_2, f_3 \in \mathcal{F}(\mathbb{R})$ and define $g = f_1 \circ (f_2 \circ f_3) \quad : \mathbb{R} \to \mathbb{R}$

$$h = (f_1 \circ f_2) \circ f_3 \quad : \mathbb{R} \to \mathbb{R}$$

$$g(x) = f_1 \circ (f_2 \circ f_3)(x) = f_1\big((f_2 \circ f_3)(x)\big) = f_1\big(f_2(f_3(x))\big)$$

$$h(x) = \big((f_1 \circ f_2) \circ f_3\big)(x) = (f_1 \circ f_2)(f_3(x)) = f_1\big(f_2(f_3(x))\big)$$

$\Longrightarrow$ $(\mathcal{F}(\mathbb{R}), \circ)$ semigroup

Algebra – Part 3

$(S, \circ)$ semigroup $\rightsquigarrow$ $e \in S$ with $e \circ a = a = a \circ e$

Definition: An element $e \in S$ is called

- left neutral (= a left identity) $\quad e \circ a = a \quad$ for all $a \in S$
- right neutral (= a right identity) $\quad a \circ e = a \quad$ for all $a \in S$
- neutral (= an identity) $\quad e \circ a = a = a \circ e \quad$ for all $a \in S$

Example: $S = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$ with $\circ$ given by the matrix multiplication

$\hookrightarrow$ $(S, \circ)$ semigroup

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ left neutral}$$

$$\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ not right neutral}$$

Fact: Let $e \in S$ be left neutral and $\tilde{e} \in S$ be right neutral.

$$\left. \begin{array}{lcl} e \circ a = a & \overset{\text{for } a = \tilde{e}}{\implies} & e \circ \tilde{e} = \tilde{e} \\ & & \quad \| \\ b \circ \tilde{e} = b & \overset{\text{for } b = e}{\implies} & e \circ \tilde{e} = e \end{array} \right\} \implies e = \tilde{e}$$

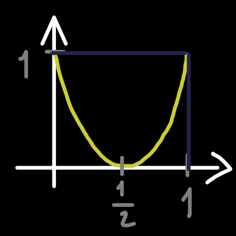
Definition: $(S, \circ)$ semigroup with identity e (the neutral element), $a, b, c \in S$.

- $x \in S$ is called a left inverse of a if $x \circ a = e$
  left invertible
- $y \in S$ is called a right inverse of b if $b \circ y = e$
  right invertible
- $z \in S$ is called an inverse of c $\quad z \circ c = e = c \circ z$
  invertible

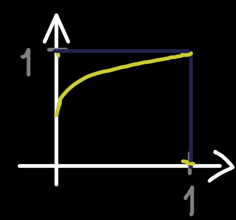
<u>Example:</u> Functions $f: [0,1] \to [0,1]$, $(\mathcal{F}([0,1]), \circ)$ semigroup

Neutral element: $\mathrm{id} : [0,1] \to [0,1]$, $x \mapsto x$

Right invertible : $\tilde{f} : [0,1] \to [0,1]$, $x \mapsto 4(x - \frac{1}{2})^2$



Right inverse of $\tilde{f}$: $g : [0,1] \to [0,1]$, $x \mapsto \frac{1}{2}\sqrt{x} + \frac{1}{2}$



$\tilde{f} \circ g = \mathrm{id}$

$g \circ \tilde{f} \neq \mathrm{id}$

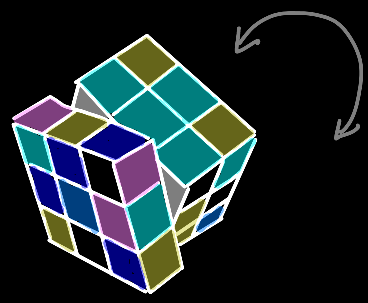
Remember:

surjective $\Longleftrightarrow$ right invertible

injective $\Longleftrightarrow$ left invertible

Algebra - Part 4

$(S, \circ)$ semigroup $\rightsquigarrow$ + neutral element, inverses $\rightsquigarrow$ group



Definition: A pair $(G, \circ)$ is called a group if:

(a) $(G, \circ)$ semigroup.

(b) There is a left identity $e \in G$.

(c) Each $a \in G$ is left invertible, i.e. there exists $b \in G$ with $b \circ a = e$.

This implies: A set G together with a binary operation $\circ$ is a group if:

(G1) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$ (associative)

(G2) There is a unique identity $e \in G$: $e \circ a = a = a \circ e$ for all $a \in G$

(G3) Each $a \in G$ is invertible: $\exists b \in G$: $b \circ a = e = a \circ b$

$a^{-1} := b$ (common notation)

Proof: (a) $\Rightarrow$ (G1) ✓

> (b) There is a left identity $e \in G$.
>
> (c) Each $a \in G$ is left invertible, i.e. there exists $b \in G$ with $\underbrace{b \circ a = e}_{(*)}$.

Let $a \in G$.

Choose $b \in G$ with $ba = e$. Then $ab \overset{(b)}{=} a(eb) \overset{(*)}{=} a(ba)b = (ab)(ab)$. (**)

Choose $c \in G$ with $c(ab) = e$ (by (c))

$$\Rightarrow \quad ab \overset{(b)}{=} e(ab) = c(ab)(ab) \overset{(**)}{=} c(ab) = e \quad \Rightarrow \text{(G3)} ✓$$

$$\Rightarrow \quad ae \overset{(*)}{=} a(ba) = (ab)a = ea = a \quad \Rightarrow \text{(G2)} ✓$$

Algebra - Part 5

Group: G together with binary operation $\circ$ and:

(G1) associativity $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$

(G2) unique identity $e \in G$: $e \circ a = a = a \circ e$ for all $a \in G$

(G3) all inverses exist: $\forall a \in G \; \exists b \in G: \; b \circ a = e = a \circ b$

$a^{-1} := b$ (common notation)

Uniqueness of inverses:

$(S, \circ)$ semigroup with identity $e \in S$.

If $a \in S$ is a left invertible with x $(x \circ a = e)$ and right invertible with y, $(a \circ y = e)$

then $x = y$.

Proof: $x = x \circ e = x \circ (a \circ y) = (x \circ a) \circ y = e \circ y = y$ $\square$

Examples: (a) $G = \{e\}$ with $e \circ e = e$, $e^{-1} = e$

(b) $G = \{e, a\}$

$\circ$	e	a
e	e	a
a	a	e

$a^{-1} = a$

(c) $(\mathbb{Z}, +)$ with identity 0 and inverses $3 + (-3) = 0$

$(\mathbb{Q} \setminus \{0\}, \cdot)$ with identity 1 and inverses $\frac{1}{4} \cdot \left(\frac{1}{4}\right)^{-1} = 1$

$(\mathbb{C}^{n \times n}, +)$ with identity $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

$(\{A \in \mathbb{C}^{n \times n} \mid \det(A) \neq 0\}, \cdot)$ with identity $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

General example: Let $(S, \circ)$ be a semigroup with identity $e \in S$.

$$S^* := \{ a \in S \mid a \text{ is invertible} \}$$

a^{-1} exists

Then $(S^*, \circ)$ is a group.

Proof: (1) $e \circ e = e \Rightarrow e \in S^*$ with $e^{-1} = e \Rightarrow$ (G2) ✓

(2) $a \in S^* \Rightarrow \begin{array}{l} a^{-1} \circ a = e \\ a \circ a^{-1} = e \end{array} \Rightarrow a^{-1} \in S^* \Rightarrow$ (G3) ✓

(3) $a, b \in S^* \Rightarrow (b^{-1} \circ a^{-1}) \circ (a \circ b) \overset{\text{associativity in } S}{=} b^{-1} \circ \underbrace{(a^{-1} \circ a)}_{e} \circ b = e$

$$(a \circ b) \circ (b^{-1} \circ a^{-1}) \overset{\text{associativity in } S}{=} a \circ (b \circ b^{-1}) \circ a^{-1} = e$$

$\Rightarrow (S^*, \circ)$ is a well-defined semigroup $\square$

Algebra - Part 6

$(S, \circ)$ semigroup. Let's write: $ab := a \circ b$

neutral element + all inverses

group

Fact: Let $(G, \circ)$ be a group and $a, b, x, y \in G$. Then:

$$ax = ay \implies x = y \quad \text{(left cancellation property)}$$

$$xb = yb \implies x = y \quad \text{(right cancellation property)}$$

Proof: $x = x\underset{\text{neutral element}}{e} = x(bb^{-1}) = (xb)b^{-1} = (yb)b^{-1} = y(bb^{-1}) = y$

Definition: $(S, \circ)$ semigroup (or group).

The order of S is the number of elements in S:

$$\mathrm{ord}(S) := \begin{cases} |S| = \# S & \text{if } S \text{ is finite} \\ \infty & \text{if } S \text{ is not finite} \end{cases}$$

Lemma: Let $(S, \circ)$ be a semigroup. Then:

$$(S, \circ) \text{ is group} \iff \forall a, b \in S \; \exists x, y \in S: \; ax = b, \; ya = b$$

Proof: $(\Rightarrow)$ Assume $(S, \circ)$ is a group. For given $a, b \in S$, set:

$$x = a^{-1}b, \quad y = ba^{-1}$$

$(\Leftarrow)$ For given $a \in S$, there are $x, y \in S$ with $ax = a$, $ya = a$.

Let's call $e := y$: $ea = a$

Let's take $b \in S$. Then there is $\tilde{x} \in S$ with $a\tilde{x} = b$.

We get: $eb = e(a\tilde{x}) = (ea)\tilde{x} = a\tilde{x} = b \implies e$ left neutral

For given $b \in S$ there is $\tilde{y} \in S$ such that: $\tilde{y}b = e \implies b$ left invertible

$\overset{\text{part 4}}{\implies} (S, \circ)$ is a group $\square$

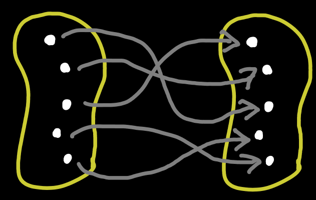
Proposition: Let $(S, \circ)$ be a semigroup with $\mathrm{ord}(S) < \infty$. Then:

$$(S, \circ) \text{ is group} \iff \text{both cancellation properties hold}$$

$$\begin{pmatrix} ax = ay & \Rightarrow & x = y \\ xb = yb & \Rightarrow & x = y \end{pmatrix}$$

Proof: For any map $f : S \to S$:

$$f \text{ is injective} \iff f \text{ is surjective}$$



For given $a \in S$, define $f_a : S \to S$ and $g_a : S \to S$ by

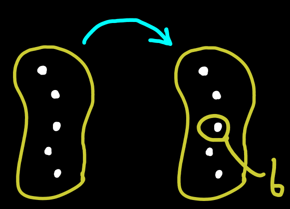
$$f_a(x) = a\,x \quad , \quad g_a(x) = x\,a \,.$$

Then we have: both cancellation properties hold

$$\iff \forall a \in S : \quad \begin{array}{l} f_a(x) = f_a(y) \Rightarrow x = y \\ g_a(x) = g_a(y) \Rightarrow x = y \end{array}$$

$$\iff \forall a \in S : \quad f_a \text{ and } g_a \text{ are injective}$$

$$\iff \forall a \in S : \quad f_a \text{ and } g_a \text{ are surjective}$$



$$\iff \forall a \in S : \quad \text{for every } b \in S \text{ there are}$$

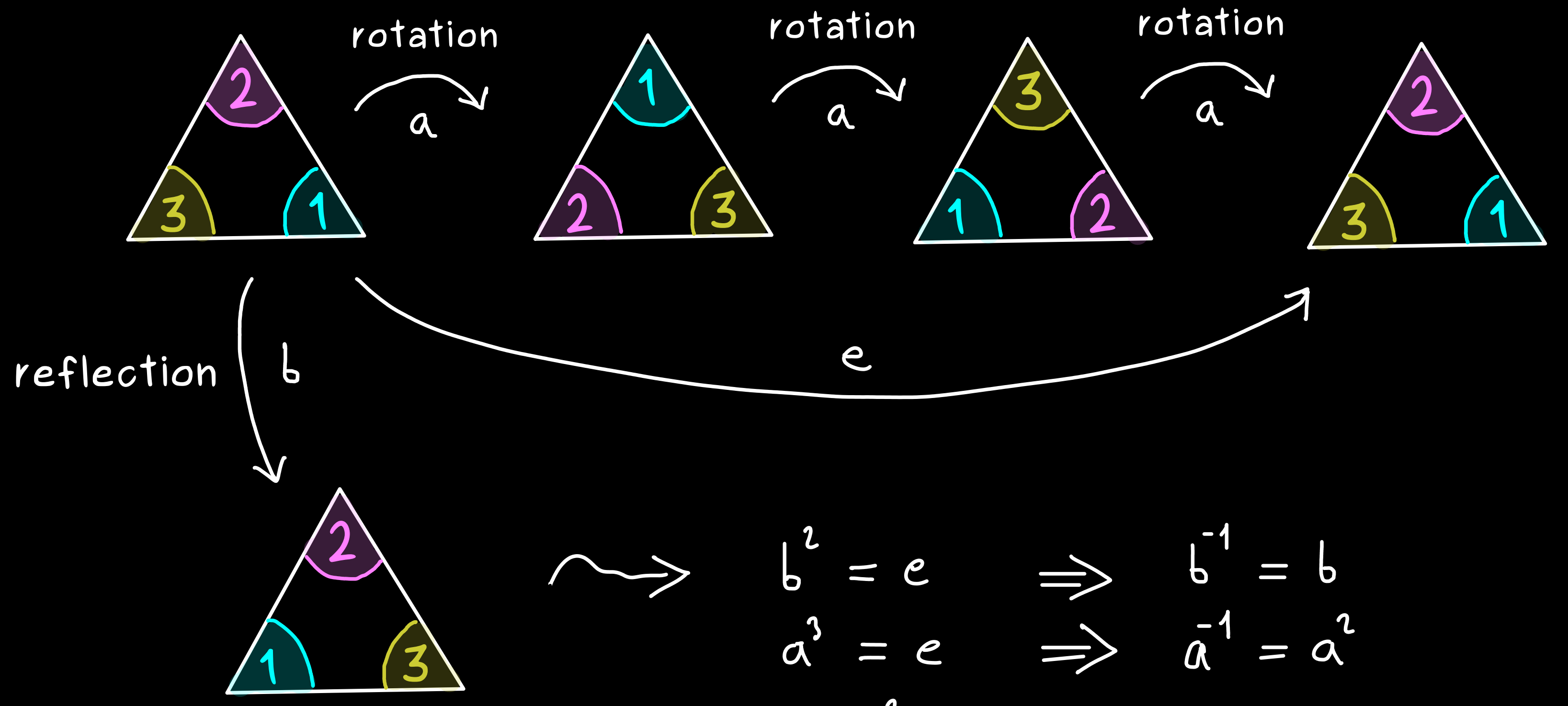
$$x, y \in S : \quad \underbrace{f_a(x)}_{=\,a\,x} = b \quad \text{and} \quad \underbrace{g_a(y)}_{=\,y\,a} = b$$

Lemma

$$\iff (S, \circ) \text{ is group} \qquad \square$$

Algebra - Part 7

Group:

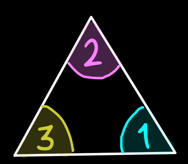


$\rightsquigarrow$ $b^2 = e \implies b^{-1} = b$

$a^3 = e \implies a^{-1} = a^2$

$ab \stackrel{?}{=} ba$

symmetry operations $\longleftrightarrow$ permutations of $\{1, 2, 3\} =: X$



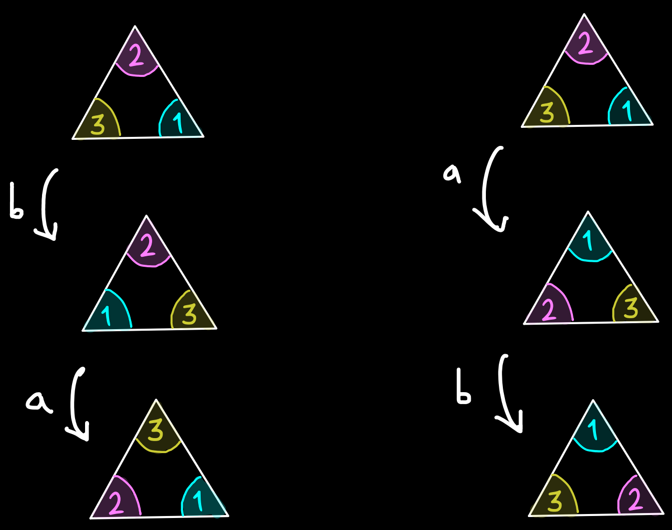
$S_3 := \{ f : X \to X \mid f \text{ bijective} \}$

$\hookrightarrow$ symmetric group

Example:

$f_b(1) = 3$	$f_a(1) = 2$
$f_b(2) = 2$	$f_a(2) = 3$
$f_b(3) = 1$	$f_a(3) = 1$

$\implies (S_3, \circ)$ composition of maps



We get:

$$(f_a \circ f_b)(1) = 1 \quad , \quad (f_b \circ f_a)(1) = 2$$
$$(f_a \circ f_b)(2) = 3 \quad , \quad (f_b \circ f_a)(2) = 1$$
$$(f_a \circ f_b)(3) = 2 \quad , \quad (f_b \circ f_a)(3) = 3$$

$\Longrightarrow$ not commutative!

<u>Definition:</u> A group $(G, \circ)$ (or semigroup) is called <u>abelian</u> or <u>commutative</u> if $a \circ b = b \circ a$ for all $a, b \in G$.

<u>Examples:</u> $(\mathbb{Z}, +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{R}, +)$, $(\mathbb{C} \setminus \{0\}, \cdot)$ are abelian.

<u>General example:</u> $G = \{a, b, e\}$

group with three elements

$\circ$	a	b	e
a	a^2	$\square$	a
b	$\square$	b^2	b
e	a	b	e

<u>1st case:</u> $a^{-1} = b$, $b^{-1} = a$ $\Rightarrow$ $a \circ b = e$, $b \circ a = e$ $\rightsquigarrow$ abelian group

<u>2nd case:</u> $a^{-1} = a$, $b^{-1} = b$ $\Rightarrow$ $(b \circ a) \circ (a \circ b) = b \circ \underbrace{a^2}_{=e} \circ b = e$

$$\Rightarrow \underbrace{(a \circ b)^{-1}}_{= a \circ b} = (b \circ a) \quad \rightsquigarrow \text{abelian group}$$

<u>Non-abelian group:</u> Symmetric group S_3 : $|S_3| = 3! = 6$ — order 6

$\quad\quad\quad\quad$ $\|$

Dihedral group D_3

Algebra - Part 8

modulus calculation:

$$13 - 12 = 1$$

modulus $= m$

$$24 - 2 \cdot 12 = 0$$

$x \sim_m y \iff$ There is $q \in \mathbb{Z}$: $x - y = q \cdot m$

$x \equiv y \pmod{m}$

Integers modulo m: $\mathbb{Z}_m$, $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Z}/m$, $\mathbb{Z}/\sim_m$

$$\mathbb{Z}_m := \{[0], [1], \dots, [m-1]\}, \qquad m \in \mathbb{N}$$

for example with $m = 12$: $[2] = \{2, 14, 26, 38, \dots, -10, -22, \dots\}$

define addition: $[k] + [\ell] := [k + \ell]$ well-defined

$[k] + [-k] = [0]$ identity, inverse

$\Rightarrow$ $(\mathbb{Z}_m, +)$ abelian group of order m

Example: $(\mathbb{Z}_2, +)$: $[0] = \{0, 2, 4, \dots, -2, -4, \dots\}$

$[1] = \{1, 3, 5, 7, \dots, -1, -3, \dots\}$

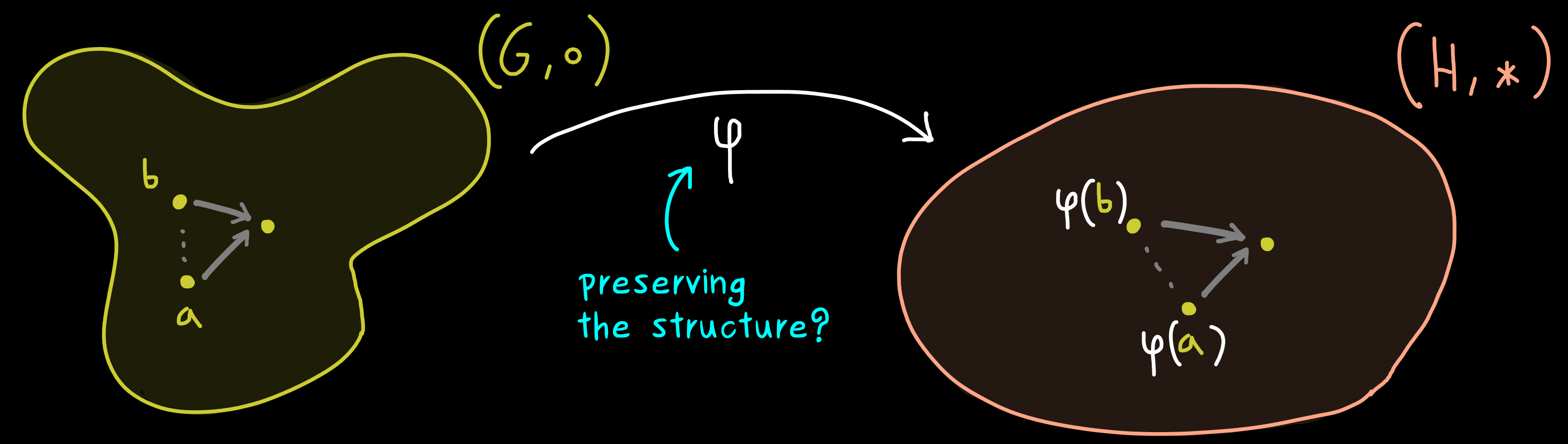
+	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

$(\mathbb{Z}_6, +)$: $[0] = \{0, 6, 12, \dots, -6, -12, \dots\}$

$[1], [2], [3], [4], [5]$

+	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]				
[2]	[2]	[3]	[4]			
[3]	[3]	[4]	[5]	[0]		
[4]	[4]	[5]	[0]	[1]	[2]	
[5]	[5]	[0]	[1]	[2]	[3]	[4]

Algebra – Part 9



$\circ$ binary operation and identity and all inverses

$*$ binary operation and identity and all inverses

Definition: $(G,\circ), (H,*)$ groups. A map $\varphi: G \rightarrow H$ is called a <u>group homomorphism</u> if $\varphi(a \circ b) = \varphi(a) * \varphi(b)$ for all $a, b \in G$.

Example: $(G,\circ) = (\mathbb{R}, +)$, $(H,*) = (\mathbb{R}\setminus\{0\}, \cdot)$.

$$\varphi: G \rightarrow H, \quad x \mapsto e^x \quad \Longrightarrow \quad \varphi(x+y) = e^{x+y} \,\rotatebox{90}{=}\, \qquad \varphi(x)\cdot\varphi(y) = e^x \cdot e^y$$

Properties: A group homomorphism satisfies:

(1) $\varphi(e_G) = e_H$ (identity is sent to identity)

(2) $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in G$.

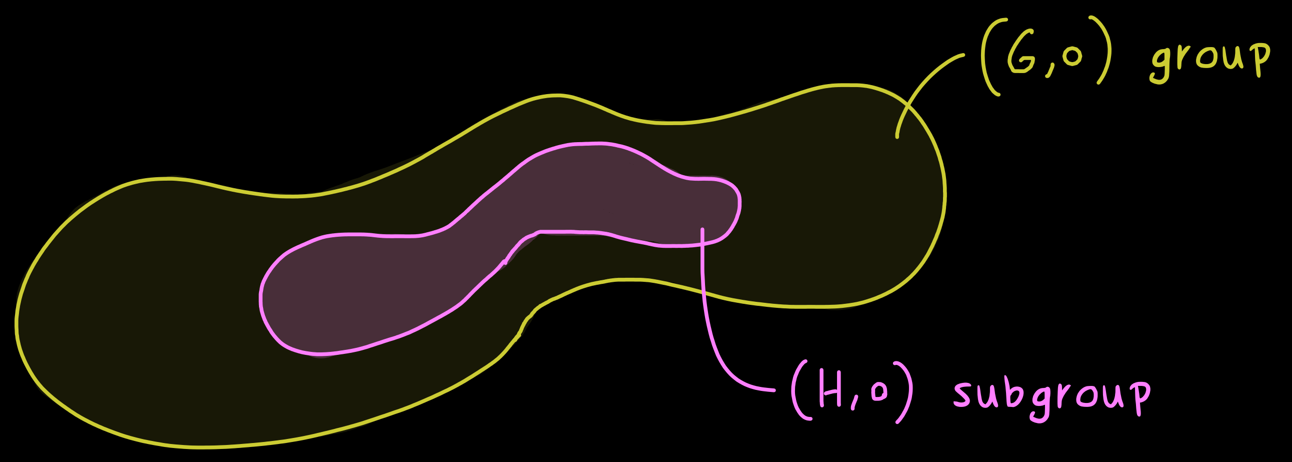
Proof: (1) $\varphi(e_G) = \varphi(e_G \circ e_G) = \varphi(e_G) * \varphi(e_G)$

$$\Longrightarrow e_H = \varphi(e_G)^{-1} * \varphi(e_G) = \varphi(e_G)^{-1} * \big(\varphi(e_G) * \varphi(e_G)\big)$$
$$= \underbrace{\big(\varphi(e_G)^{-1} * \varphi(e_G)\big)}_{= e_H} * \varphi(e_G) = \varphi(e_G)$$

(2) $e_H = \varphi(e_G) = \varphi(a^{-1} \circ a) = \varphi(a^{-1}) * \varphi(a)$

$$\overset{\text{inverse unique}}{\Longrightarrow} \varphi(a)^{-1} = \varphi(a^{-1}) \qquad \square$$

Algebra - Part 10

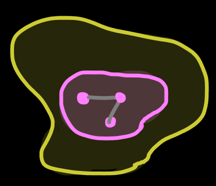


Example: $(\mathbb{R},+) \leftarrow (\mathbb{Z},+)$ subgroup

Definition: Let $(G,\circ)$ be a group. A non-empty subset $H \subseteq G$ is called a subgroup of G if $(H,\circ)$ forms a group.

We get a group homomorphism: $\varphi: H \to G,\ x \mapsto x$, $\quad \varphi(a\circ b) = \varphi(a)\circ\varphi(b)$

$$\Rightarrow \underbrace{\varphi(e_H)}_{= e_H} = e_G$$



Proposition: Let $(G,\circ)$ be a group and $H \subseteq G$ be a non-empty subset.

Then: H is a subgroup of G $\Leftrightarrow \begin{cases} a\circ b \in H & \text{for all } a,b\in H \quad (*) \\ a^{-1} \in H & \text{for all } a\in H \quad (**) \end{cases}$

Proof: $(\Rightarrow)$ Assume $(H,\circ)$ form a group.

$\Rightarrow\ \circ: H\times H \to H$ is well-defined! $\Rightarrow (*)$ ✓

Neutral element in H is the same as the neutral element in G:

$e = x^{-1}\circ x \overset{\text{inverses are unique}}{\Longrightarrow} x^{-1} \in H$ for all $x\in H \Rightarrow (**)$ ✓

$(\Leftarrow)$ Assume $(*)$, $(**)$. Since $a\circ b\in H$ for all $a,b\in H$,

$\circ: H\times H\to H$ is well-defined!

associative! (G is a group)

Choose $a\in H \overset{(**)}{\Longrightarrow} a^{-1}\in H \overset{(*)}{\Longrightarrow} a\circ a^{-1} = e \in H$

$\Rightarrow (H,\circ)$ is a group $\square$

Example: (a) $(G,\circ)$ group. $\left.\begin{array}{l}\{e\} \text{ is subgroup of } G \\ G \text{ is subgroup of } G\end{array}\right\}$ trivial subgroups

(b) $(\mathbb{Z},+)$ group, $m \in \mathbb{N}$. $m\mathbb{Z} := \{m \cdot k \mid k \in \mathbb{Z}\} \subseteq \mathbb{Z}$

$$\Longrightarrow \quad (m\mathbb{Z},+) \text{ subgroup of } (\mathbb{Z},+)$$

Recall: $\mathbb{Z}/m\mathbb{Z}$ is a group $\rightsquigarrow$ general construction G/H

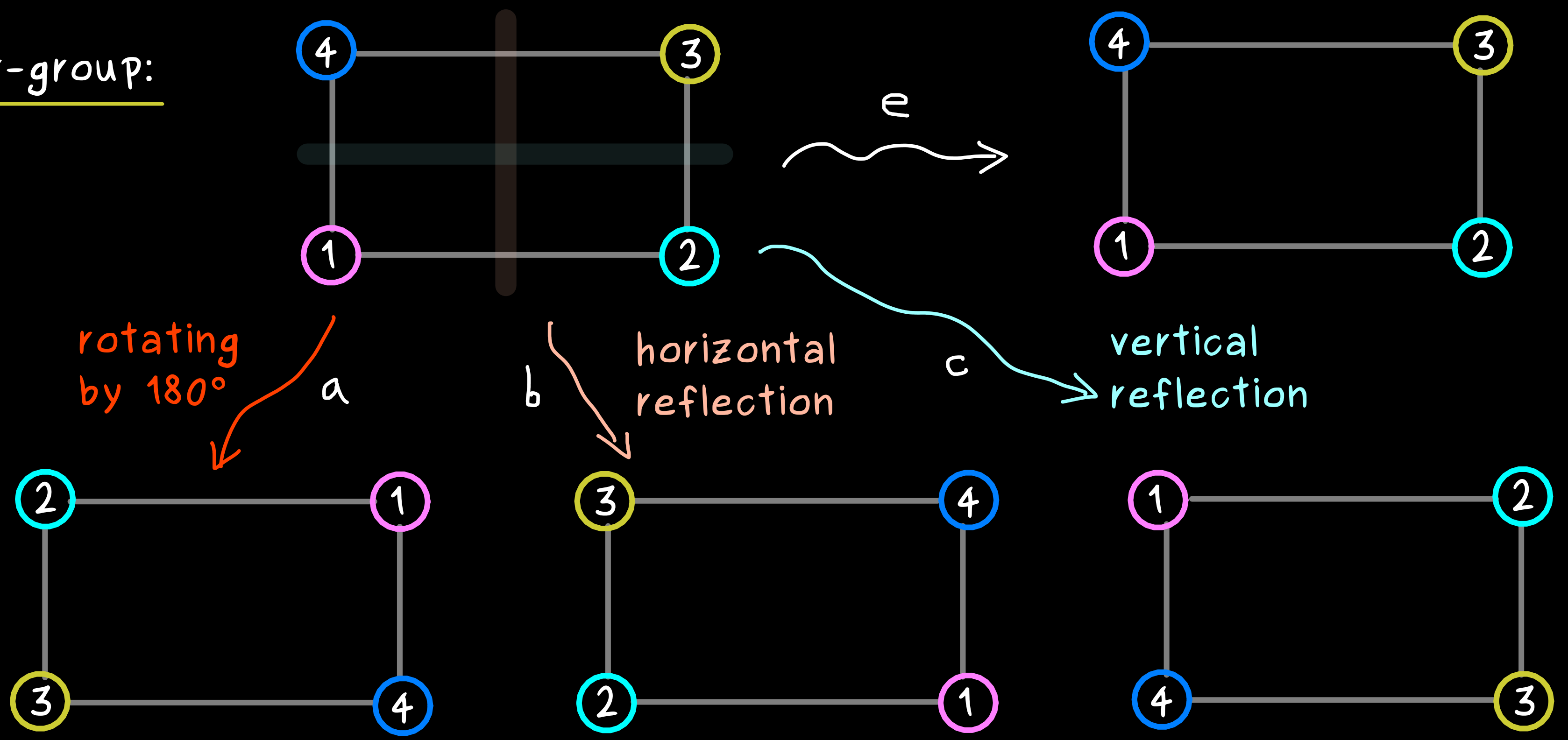
Algebra - Part 11

Recall subgroups: $(G,\circ) \rightsquigarrow H \subseteq G,\ (H,\circ)$ group $\rightsquigarrow H$ subgroup of G $\rightsquigarrow H \leq G$

Proposition: $(G,\circ)$ group, $H \subseteq G$ non-empty subset.

$$H \leq G \iff \begin{cases} a \circ b \in H & \text{for all } a,b \in H \\ a^{-1} \in H & \text{for all } a \in H \end{cases}$$

Klein four-group:



$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\rightsquigarrow$ associativity ✓

$(G,\circ)$ with $G = \{e,a,b,c\}$ and $\circ$ satisfying the table above defines the so-called Klein four group, called K_4.

Proposition: Let $(G,\circ)$ be a group with $\mathrm{ord}(G) < \infty$, $H \subseteq G$ be a non-empty subset.

Then: $H \leq G \iff a \circ b \in H$ for all $a, b \in H$

Proof: $(\Rightarrow)$ ✓ $(\Leftarrow)$ $(H,\circ)$ semigroup of finite order and both cancellation properties hold

$$\begin{pmatrix} a \circ x = a \circ y \Rightarrow x = y \\ x \circ b = y \circ b \Rightarrow x = y \end{pmatrix}$$

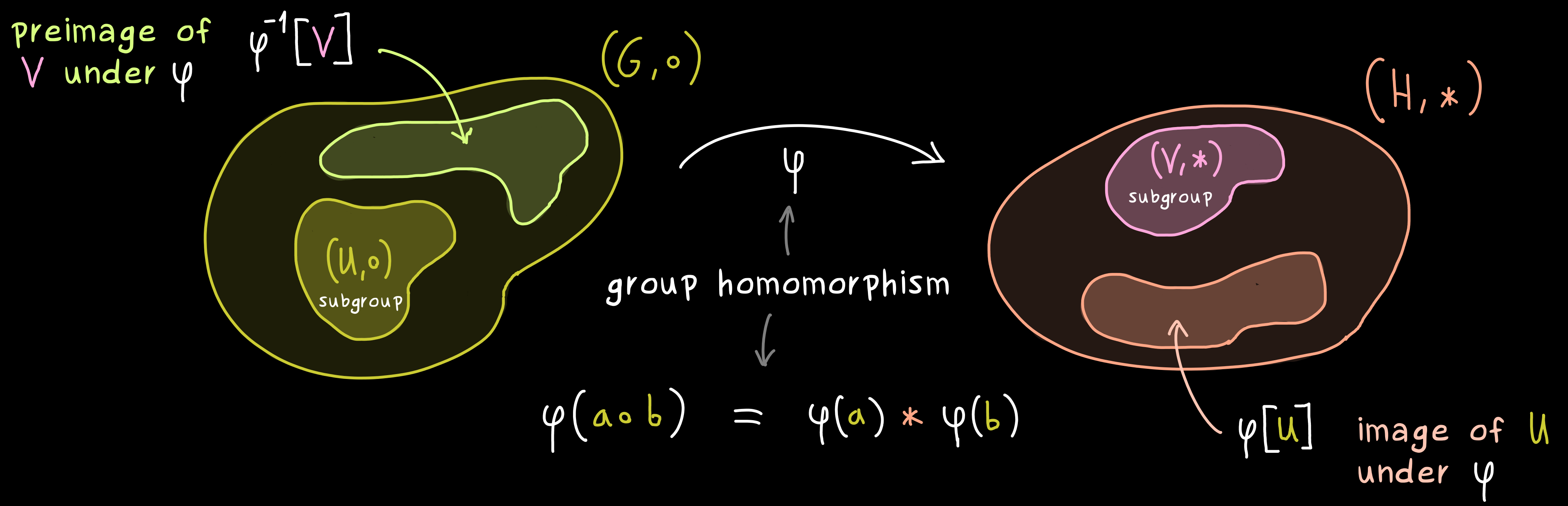
$$\overset{\text{part 6}}{\Longrightarrow} \quad (H,\circ) \text{ is a group} \qquad \square$$

Example: $G = \{e, a, b, c\}$ Klein four-group.

subgroups: $H_1 = \{e\}$, $H_2 = \{e, a\}$, $H_3 = \{e, b\}$, $H_4 = \{e, c\}$, $H_5 = G$

$\rightsquigarrow$ we have 5 subgroups

Algebra - Part 12



Proposition: $(G,\circ)$, $(H,*)$ groups, $\varphi: G \longrightarrow H$ group homomorphism.

If $U \subseteq G$ is a subgroup of G and $V \subseteq H$ is a subgroup of H,

then: (a) $\varphi[U] \subseteq H$ is a subgroup of H

(b) $\varphi^{-1}[V] \subseteq G$ is a subgroup of G

Proof: (a) Take $a, b \in \varphi[U] \subseteq H$. We find $x, y \in U$ with $\varphi(x) = a$, $\varphi(y) = b$.

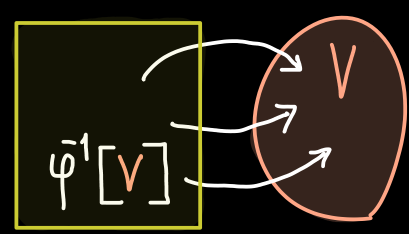
Then: $a * b = \varphi(x) * \varphi(y) = \varphi(\underbrace{x \circ y}_{\in U \text{ (subgroup!)}}) \in \varphi[U]$

$a^{-1} = \varphi(x)^{-1} = \varphi(\underbrace{x^{-1}}_{\in U \text{ (subgroup!)}}) \in \varphi[U]$

$\overset{\text{part 10}}{\Longrightarrow} (\varphi[U], *)$ subgroup

(b) Take $x, y \in \varphi^{-1}[V]$. We find $a, b \in V$ with $\varphi(x) = a$, $\varphi(y) = b$.

Then: $\varphi(x \circ y) = \varphi(x) * \varphi(y) = a * b \in V$



$\Longrightarrow x \circ y \in \varphi^{-1}[V]$

$\varphi(x^{-1}) = \varphi(x)^{-1} = a^{-1} \in V$

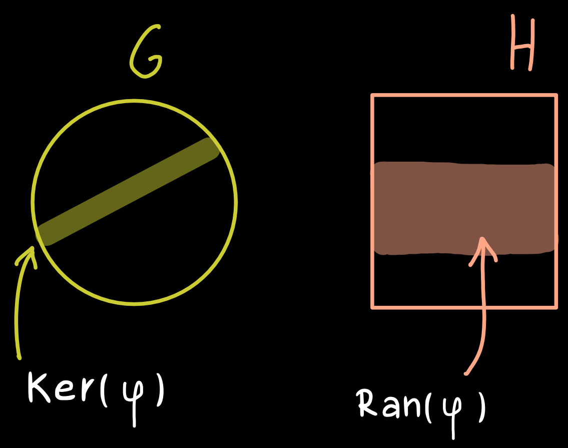
$\Longrightarrow x^{-1} \in \varphi^{-1}[V] \overset{\text{part 10}}{\Longrightarrow} (\varphi^{-1}[V], \circ)$ subgroup $\square$

Special cases: $\varphi: G \longrightarrow H$ group homomorphism.

$$\varphi^{-1}[\{e\}] =: \mathrm{Ker}(\varphi) \quad \text{kernel of } \varphi$$

$$\varphi[G] =: \mathrm{Ran}(\varphi) \quad \text{range of } \varphi$$

$$\left(\mathrm{im}(\varphi) \quad \text{image of } \varphi\right)$$



Example: $\varphi: \mathbb{Z} \longrightarrow \{e, a\}$

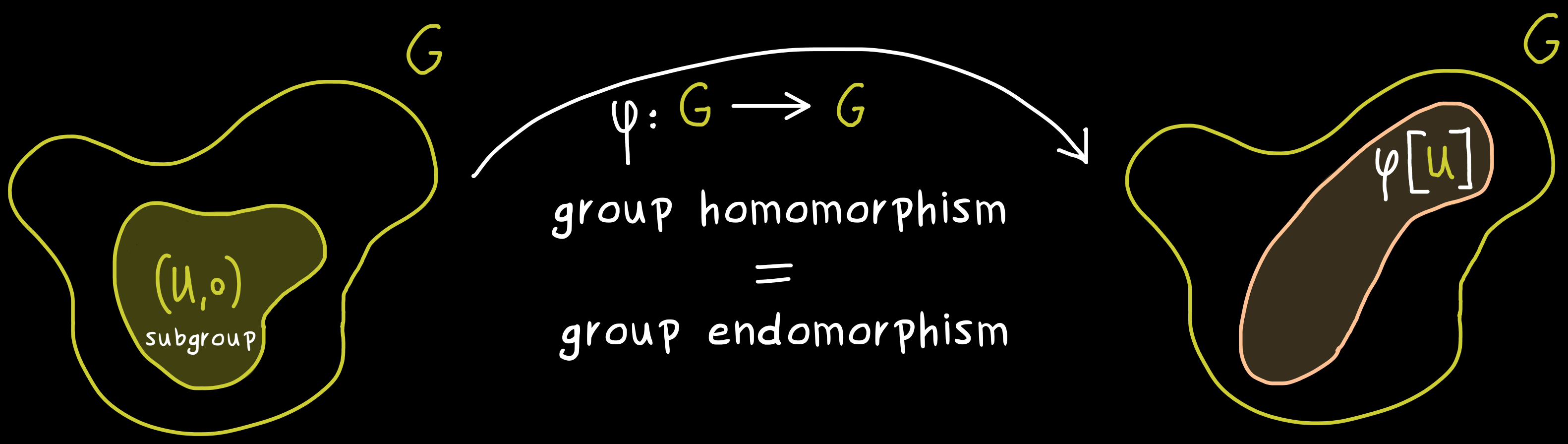
$$k \longmapsto \begin{cases} e, & k \text{ even} \\ a, & k \text{ odd} \end{cases}$$

$\rightsquigarrow$ group homomorphism!

$$\varphi(k+m) = \varphi(k) \circ \varphi(m)$$

$$\mathrm{Ker}(\varphi) = \{\text{even numbers}\} = 2 \cdot \mathbb{Z} \qquad \text{subgroup!}$$

Algebra - Part 13



Important case: $\underbrace{\text{inner}}$ $\underbrace{\text{automorphisms}}$: $\varphi: G \to G$ group homomorphism that can be written as $\varphi(x) = g\,x\,g^{-1}$

φ is represented by an inner element

endomorphism + isomorphism

= (bijective and homomorphism in both directions)

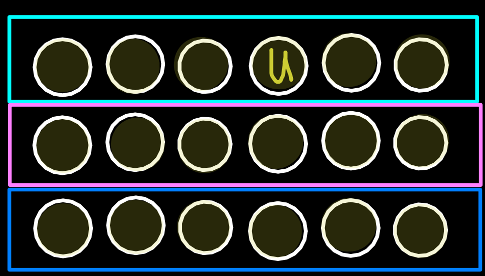
We already know: $U \subseteq G$ subgroup $\Longrightarrow$ $\varphi[U]$, $\varphi^{-1}[U]$ subgroups

Definition: Two subgroups $U, V \subseteq G$ are called conjugate subgroups if there is an element $g \in G$: $V = gUg^{-1} := \{ g u g^{-1} \mid u \in U \}$

$= \varphi[U]$ for $\varphi: G \to G$, $\varphi(x) = g\,x\,g^{-1}$

Remember: This defines an equivalence relation on the set of subgroups of G.

Hence:



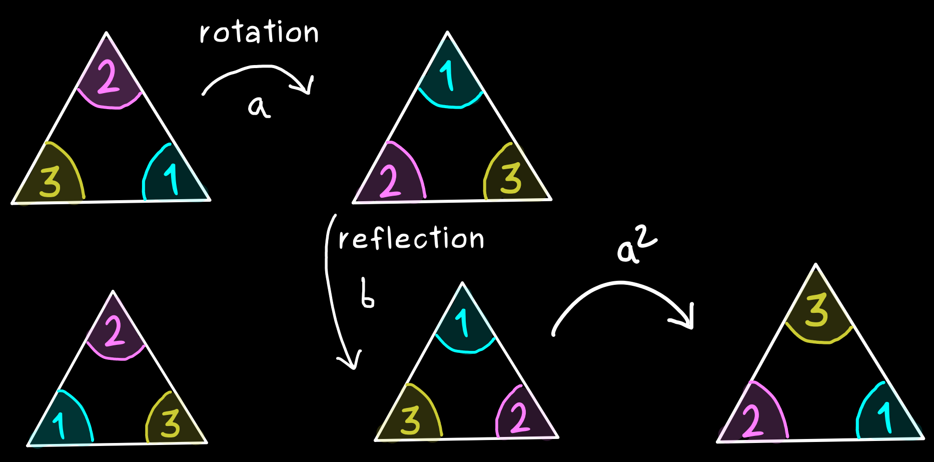
$[U] := \{ gUg^{-1} \mid g \in G \}$

equivalence class

Trivial for abelian groups: $gUg^{-1} = \{ u g g^{-1} \mid u \in U \} = U$

Example: Symmetric group S_3

$$S_3 = \{e, a, b, a^2, ab, ba\}$$



$$U = \{e, b\} \xrightarrow{\text{conjugate subgroups}}$$

$$aUa^{-1} = \{e, \underbrace{aba^2}_{ba}\} = \{e, ba\}$$

$$a^2U(a^2)^{-1} = \{e, \underbrace{a^2ba}_{ab}\} = \{e, ab\}$$

$$abU(ab)^{-1} = \{e, a\underbrace{bb}_{=e}(ab)\} = \{e, ba\}$$

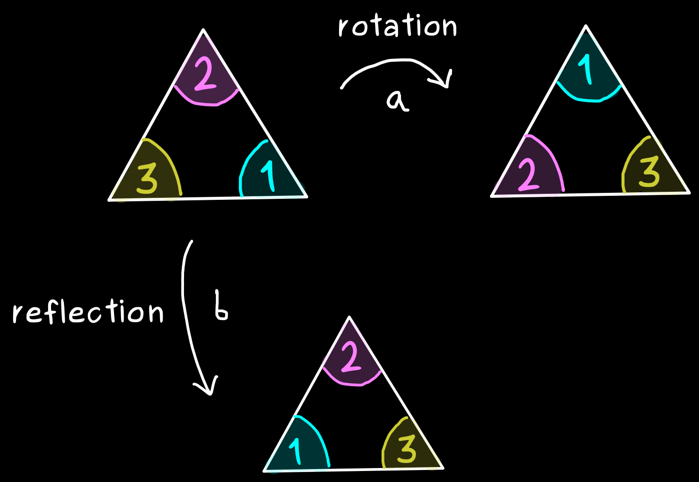
$$baU(ba)^{-1} = \{e, ba\underbrace{b(b}_{e}a)\} = \{e, ab\}$$

$$bUb^{-1} = eUe^{-1} = U$$

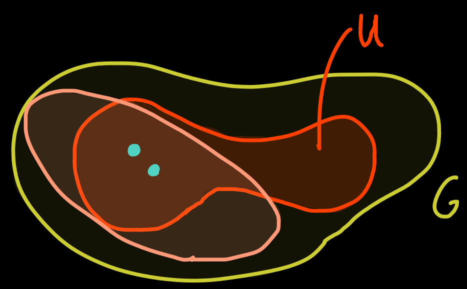
Algebra - Part 14

Recall: Symmetric group S_3

is generated by the two elements a, b



Definition: Let G be a group and $S \subseteq G$ be a subset.



$$\langle S \rangle := \bigcap_{\substack{U \subseteq G \text{ subgroup} \\ \text{with } S \subseteq U}} U$$

We say: S generates the subgroup $\langle S \rangle$.

Proposition: Intersection of subgroups is also a subgroup.

Proof: Assume: G group, $U_j \subseteq G$ subgroups for all $j \in J$, $\tilde{U} := \bigcap_{j \in J} U_j$.

Obvious: $e \in \tilde{U}$ ✓

Take $a, b \in \tilde{U} \Longrightarrow a, b \in U_j$ for all $j \in J$

$\overset{U_j \text{ subgroup}}{\Longrightarrow} \; ab \in U_j$ and $a^{-1} \in U_j$ for all $j \in J$

$\Longrightarrow \; ab \in \tilde{U}$ and $a^{-1} \in \tilde{U}$ $\square$

Fact: If $S \neq \emptyset$ and $S^{-1} := \{ s^{-1} \mid s \in S \}$, then:

$$\langle S \rangle = \{ a_1 a_2 \cdots a_n \in G \mid n \in \mathbb{N},\ a_1, \dots, a_n \in S \cup S^{-1} \}$$

Example: Symmetric group S_3: $S = \{a, b\}$, $S^{-1} = \{a^2, b\}$

$\rightsquigarrow$ ab , ba , $bb = e$, ... just six elements

$$S_3 = \langle a, b \rangle$$

Definition: A group G is called cyclic if there is element $g \in G$ such that $\langle g \rangle = G$.

In other words: $G = \{ g^k \mid k \in \mathbb{Z} \}$ with $g^0 :=$ identity element in G

Algebra - Part 15

Cyclic group: $G = \langle g \rangle$ for a particular $g \in G$

$= \{ g^k \mid k \in \mathbb{Z} \}$ with $g^0 :=$ identity element in G

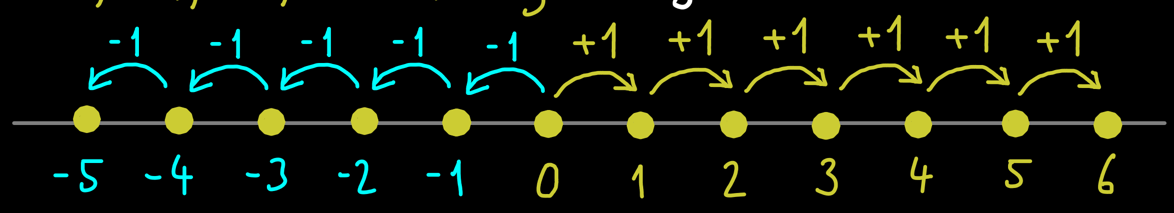
always abelian: $g^k g^m = g \cdot g \cdots g \cdot g \cdot g \cdots g = g^{k+m} = g^m g^k$

Examples:

(a) $G = \{e\}$, $G = \langle e \rangle$ $\quad (G = \langle \emptyset \rangle)$

(b) $\mathbb{Z} = \{ \dots, -2, -1, 0, 1, 2, \dots \}$ together with addition $+$

We know:



$\mathbb{Z} = \langle 1 \rangle = \{ \underbrace{1 + 1 + \dots + 1}_{k \text{ times}} \mid k \in \mathbb{Z} \} = \{ k \cdot 1 \mid k \in \mathbb{Z} \}$

cyclic group! $\mathbb{Z} = \langle -1 \rangle$

(c) subgroups of $(\mathbb{Z}, +)$: $m \in \mathbb{N}$

$m\mathbb{Z} := \{ m \cdot k \mid k \in \mathbb{Z} \} \subseteq \mathbb{Z}$, $3\mathbb{Z} = \{ \dots, -6, -3, 0, 3, 6, \dots \}$

also cyclic: $m\mathbb{Z} = \langle m \rangle$

(d) $\mathbb{Z}/m\mathbb{Z}$ is a finite abelian group! $\mathbb{Z}/3\mathbb{Z} = \{ [0], [1], [2] \}$

↳ addition $[k] + [1] = [k+1]$

$\mathbb{Z}/m\mathbb{Z} = \langle [1] \rangle$ cyclic!

Important Result: For each natural number $m \in \mathbb{N}$ or $m = \infty$, there is a cyclic group of order m.