Algebra - Part 15

Cyclic group: $G = \langle g \rangle$ for a particular $g \in G$

$= \{ g^k \mid k \in \mathbb{Z} \}$ with $g^0 :=$ identity element in G

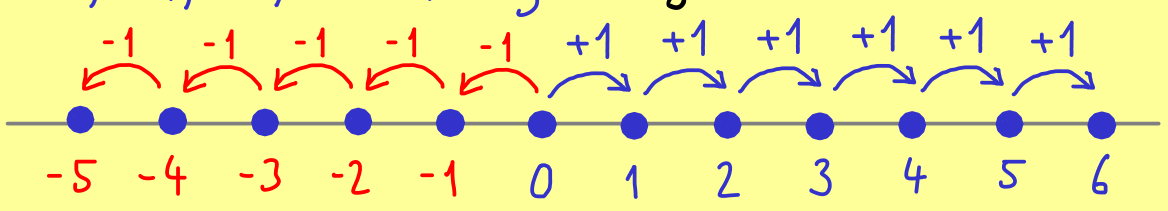
always abelian: $g^k g^m = g \cdot g \cdots g \cdot g \cdot g \cdots g = g^{k+m} = g^m g^k$

Examples:

(a) $G = \{ e \}$, $G = \langle e \rangle$ $\quad \left(G = \langle \emptyset \rangle \right)$

(b) $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$ together with addition $+$

We know:



$\mathbb{Z} = \langle 1 \rangle = \{ \underbrace{1 + 1 + \cdots + 1}_{k \text{ times}} \mid k \in \mathbb{Z} \} = \{ k \cdot 1 \mid k \in \mathbb{Z} \}$

cyclic group! $\mathbb{Z} = \langle -1 \rangle$

(c) subgroups of $(\mathbb{Z}, +)$: $m \in \mathbb{N}$

$m\mathbb{Z} := \{ m \cdot k \mid k \in \mathbb{Z} \} \subseteq \mathbb{Z}$, $3\mathbb{Z} = \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$

also cyclic: $m\mathbb{Z} = \langle m \rangle$

(d) $\mathbb{Z}/_{m\mathbb{Z}}$ is a finite abelian group! $\mathbb{Z}/_{3\mathbb{Z}} = \{ [0], [1], [2] \}$

addition $[k] + [1] = [k+1]$

$\mathbb{Z}/_{m\mathbb{Z}} = \langle [1] \rangle$ cyclic!

Important Result: For each natural number $m \in \mathbb{N}$ or $m = \infty$, there is a cyclic group of order m.