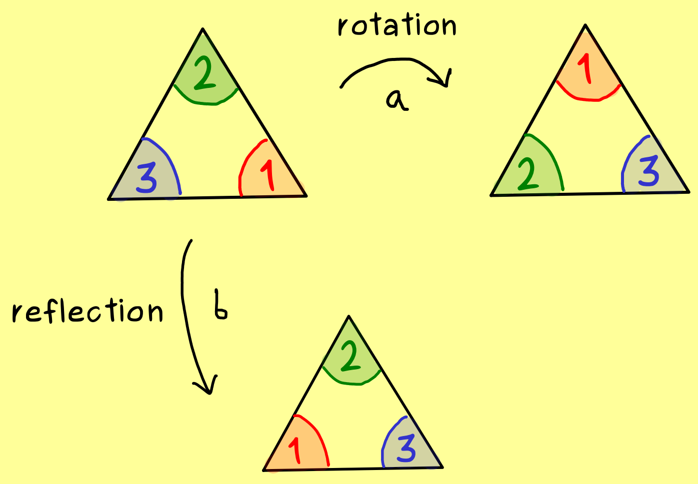
The Bright Side of Mathematics

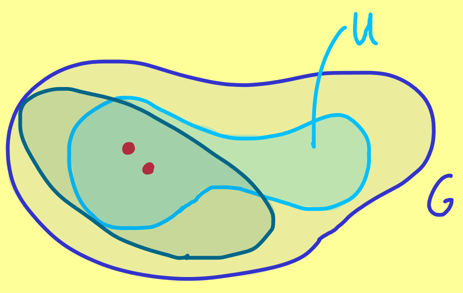
Algebra - Part 14

Recall: Symmetric group S_3

is generated by the two elements a, b



Definition: Let G be a group and $S \subseteq G$ be a subset.



$$\langle S \rangle := \bigcap_{\substack{U \subseteq G \text{ subgroup} \\ \text{with } S \subseteq U}} U$$

We say: S generates the subgroup $\langle S \rangle$.

Proposition: Intersection of subgroups is also a subgroup.

Proof: Assume: G group, $U_j \subseteq G$ subgroups for all $j \in J$, $\tilde{U} := \bigcap_{j \in J} U_j$.

Obvious: $e \in \tilde{U}$ ✓

Take $a, b \in \tilde{U} \implies a, b \in U_j$ for all $j \in J$

$\overset{U_j \text{ subgroup}}{\implies}$ $ab \in U_j$ and $a^{-1} \in U_j$ for all $j \in J$

$\implies$ $ab \in \tilde{U}$ and $a^{-1} \in \tilde{U}$ $\square$

Fact: If $S \neq \emptyset$ and $S^{-1} := \{ s^{-1} \mid s \in S \}$, then:

$$\langle S \rangle = \{ a_1 a_2 \cdots a_n \in G \mid n \in \mathbb{N},\ a_1, \dots, a_n \in S \cup S^{-1} \}$$

Example: Symmetric group S_3: $S = \{a, b\}$, $S^{-1} = \{a^2, b\}$

$\rightsquigarrow$ ab, ba, $bb = e$, ... just six elements

$$S_3 = \langle a, b \rangle$$

Definition: A group G is called cyclic if there is element $g \in G$ such that $\langle g \rangle = G$.

In other words: $G = \{ g^k \mid k \in \mathbb{Z} \}$ with $g^0 :=$ identity element in G