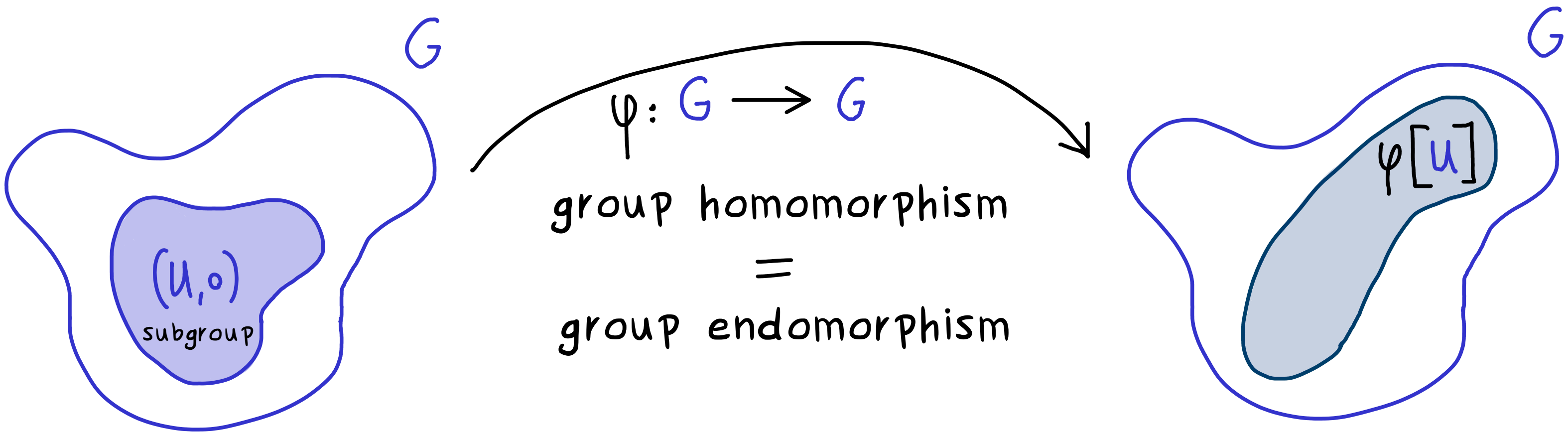
Algebra - Part 13



Important case: $\underbrace{\text{inner}}_{\text{φ is represented by an inner element}}$ $\underbrace{\text{automorphisms}}_{\text{endomorphism + isomorphism}}$: $\varphi: G \to G$ group homomorphism that can be written as $\varphi(x) = g\,x\,g^{-1}$

isomorphism = (bijective and homomorphism in both directions)

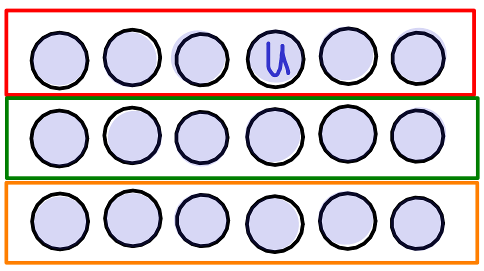
We already know: $U \subseteq G$ subgroup $\Rightarrow$ $\varphi[U]$, $\varphi^{-1}[U]$ subgroups

Definition: Two subgroups $U, V \subseteq G$ are called conjugate subgroups if there is an element $g \in G$: $V = gUg^{-1} := \{ g\,u\,g^{-1} \mid u \in U \}$

$$= \varphi[U] \quad \text{for } \varphi: G \to G, \quad \varphi(x) = g\,x\,g^{-1}$$

Remember: This defines an equivalence relation on the set of subgroups of G.

Hence:

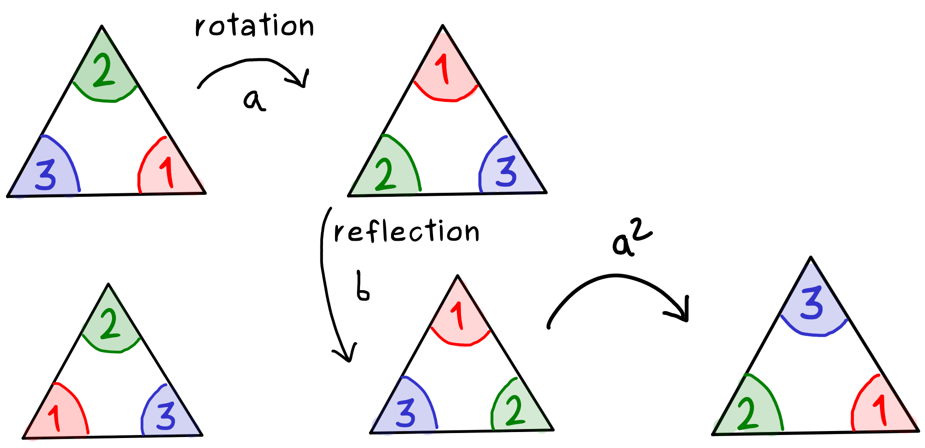


$[U] := \{ gUg^{-1} \mid g \in G \}$ ← equivalence class

Trivial for abelian groups: $gUg^{-1} = \{ u\,g\,g^{-1} \mid u \in U \} = U$

Example: Symmetric group S_3

$$S_3 = \{e, a, b, a^2, ab, ba\}$$



$$U = \{e, b\} \xrightarrow{\text{conjugate subgroups}}$$

$$aUa^{-1} = \{e, \underbrace{aba^2}_{ba}\} = \{e, ba\}$$

$$a^2U(a^2)^{-1} = \{e, \underbrace{a^2ba}_{ab}\} = \{e, ab\}$$

$$abU(ab)^{-1} = \{e, a\underbrace{bb}_{=e}(ab)\} = \{e, ba\}$$

$$baU(ba)^{-1} = \{e, ba\underbrace{b(b}_{e}a)\} = \{e, ab\}$$

$$bUb^{-1} = eUe^{-1} = U$$