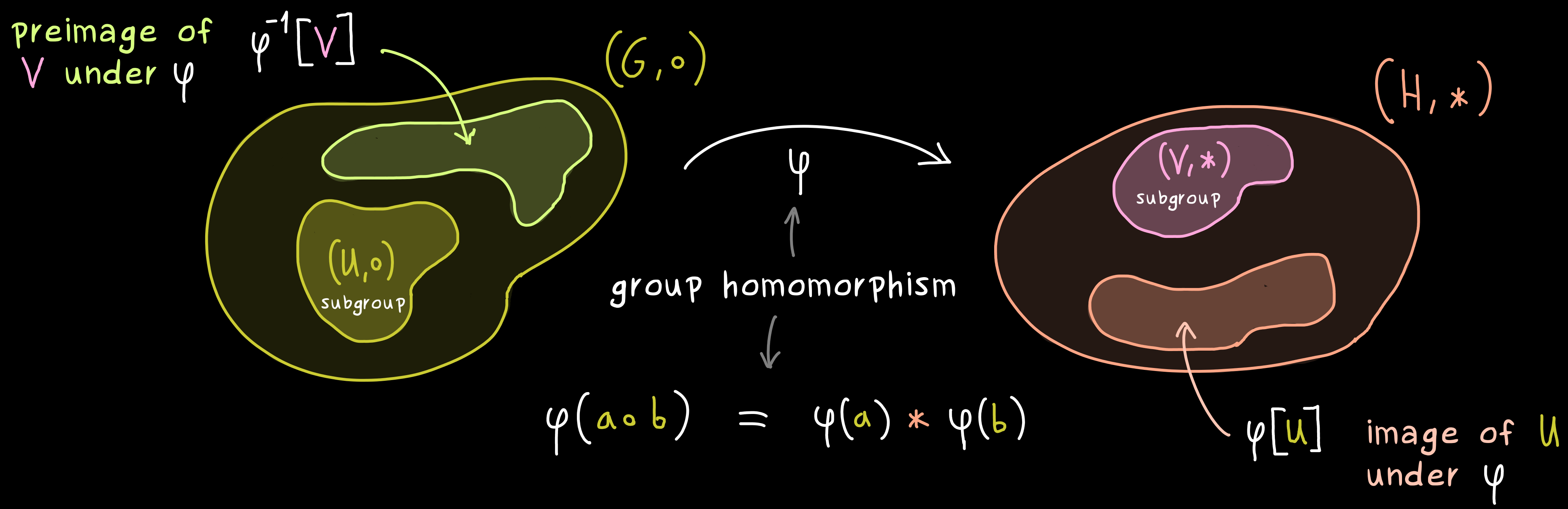
Algebra - Part 12



$$\varphi(a\circ b) = \varphi(a) * \varphi(b)$$

Proposition: $(G,\circ), (H,*)$ groups, $\varphi: G \longrightarrow H$ group homomorphism.

If $U \subseteq G$ is a subgroup of G and $V \subseteq H$ is a subgroup of H,

then:

(a) $\varphi[U] \subseteq H$ is a subgroup of H

(b) $\varphi^{-1}[V] \subseteq G$ is a subgroup of G

Proof: (a) Take $a,b \in \varphi[U] \subseteq H$. We find $x,y \in U$ with $\varphi(x) = a$, $\varphi(y) = b$.

Then: $a * b = \varphi(x) * \varphi(y) = \varphi(\underbrace{x\circ y}_{\in U \text{ (subgroup!)}}) \in \varphi[U]$

$a^{-1} = \varphi(x)^{-1} = \varphi(\underbrace{x^{-1}}_{\in U \text{ (subgroup!)}}) \in \varphi[U]$

$\overset{\text{part 10}}{\Longrightarrow}$ $(\varphi[U], *)$ subgroup

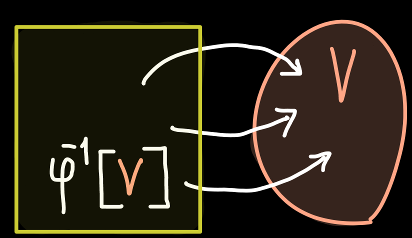
(b) Take $x,y \in \varphi^{-1}[V]$. We find $a,b \in V$ with $\varphi(x) = a$, $\varphi(y) = b$.

Then: $\varphi(x\circ y) = \varphi(x) * \varphi(y) = a * b \in V$

$\Longrightarrow$ $x\circ y \in \varphi^{-1}[V]$

$\varphi(x^{-1}) = \varphi(x)^{-1} = a^{-1} \in V$

$\Longrightarrow$ $x^{-1} \in \varphi^{-1}[V]$ $\overset{\text{part 10}}{\Longrightarrow}$ $(\varphi^{-1}[V], \circ)$ subgroup $\square$

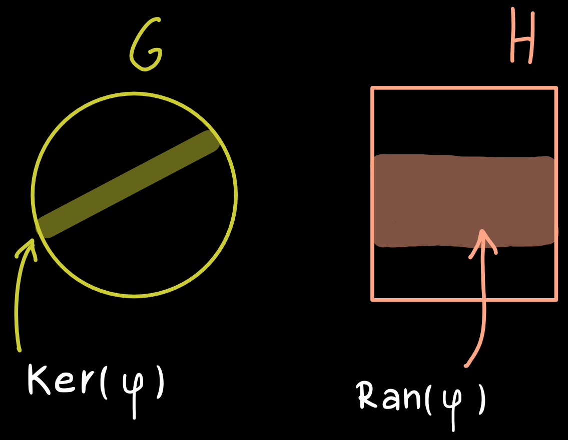


Special cases: $\varphi : G \longrightarrow H$ group homomorphism.

$$\varphi^{-1}[\{e\}] =: \mathrm{Ker}(\varphi) \quad \text{kernel of } \varphi$$

$$\varphi[G] =: \mathrm{Ran}(\varphi) \quad \text{range of } \varphi$$

$$(\mathrm{im}(\varphi) \quad \text{image of } \varphi)$$



Example: $\varphi : \mathbb{Z} \longrightarrow \{e, a\}$

$$k \longmapsto \begin{cases} e, & k \text{ even} \\ a, & k \text{ odd} \end{cases}$$

$\rightsquigarrow$ group homomorphism!

$$\varphi(k+m) = \varphi(k) \circ \varphi(m)$$

$$\mathrm{Ker}(\varphi) = \{\text{even numbers}\} = 2 \cdot \mathbb{Z} \qquad \text{subgroup!}$$