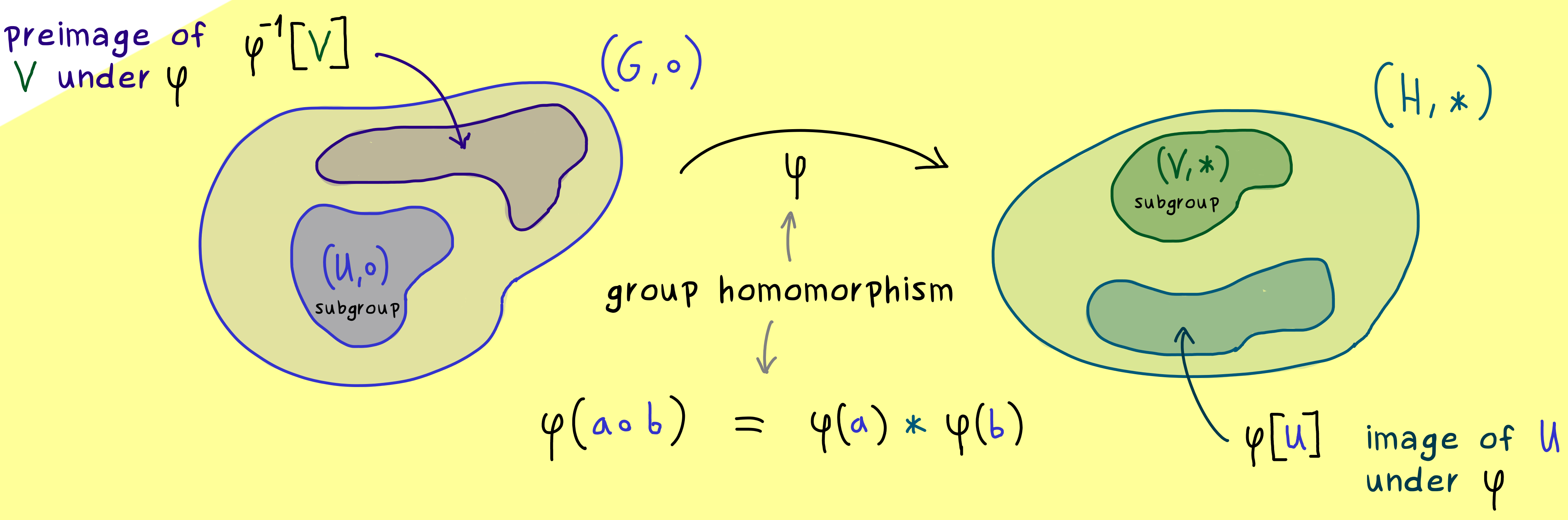
The Bright Side of Mathematics

Algebra - Part 12



$$\varphi(a\circ b) = \varphi(a) * \varphi(b)$$

Proposition: $(G,\circ),(H,*)$ groups, $\varphi: G \longrightarrow H$ group homomorphism.

If $U \subseteq G$ is a subgroup of G and $V \subseteq H$ is a subgroup of H,

then: (a) $\varphi[U] \subseteq H$ is a subgroup of H

(b) $\varphi^{-1}[V] \subseteq G$ is a subgroup of G

Proof: (a) Take $a,b \in \varphi[U] \subseteq H$. We find $x,y \in U$ with $\varphi(x) = a$, $\varphi(y) = b$.

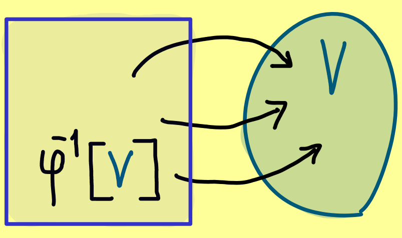
Then: $a * b = \varphi(x) * \varphi(y) = \varphi(\underbrace{x\circ y}_{\in U \text{ (subgroup!)}}) \in \varphi[U]$

$a^{-1} = \varphi(x)^{-1} = \varphi(\underbrace{x^{-1}}_{\in U \text{ (subgroup!)}}) \in \varphi[U]$ $\overset{\text{part 10}}{\Longrightarrow}$ $(\varphi[U], *)$ subgroup

(b) Take $x,y \in \varphi^{-1}[V]$. We find $a,b \in V$ with $\varphi(x) = a$, $\varphi(y) = b$.

Then: $\varphi(x\circ y) = \varphi(x) * \varphi(y) = a * b \in V$

$\Longrightarrow x \circ y \in \varphi^{-1}[V]$



$\varphi(x^{-1}) = \varphi(x)^{-1} = a^{-1} \in V$

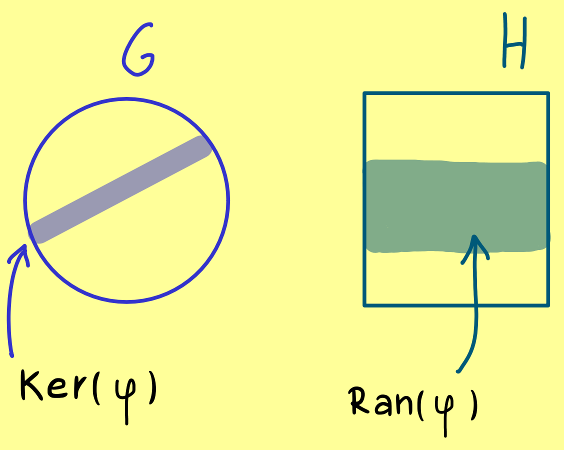
$\Longrightarrow x^{-1} \in \varphi^{-1}[V] \overset{\text{part 10}}{\Longrightarrow} (\varphi^{-1}[V], \circ)$ subgroup $\square$

Special cases: $\varphi: G \longrightarrow H$ group homomorphism.

$\varphi^{-1}[\{e\}] =: \mathrm{Ker}(\varphi)$ kernel of φ

$\varphi[G] =: \mathrm{Ran}(\varphi)$ range of φ

($\mathrm{im}(\varphi)$ image of φ)



Example: $\varphi: \mathbb{Z} \longrightarrow \{e, a\}$

$k \longmapsto \begin{cases} e, & k \text{ even} \\ a, & k \text{ odd} \end{cases}$ $\rightsquigarrow$ group homomorphism! $\varphi(k+m) = \varphi(k)\circ\varphi(m)$

$\mathrm{Ker}(\varphi) = \{\text{even numbers}\} = 2\mathbb{Z}$ subgroup!