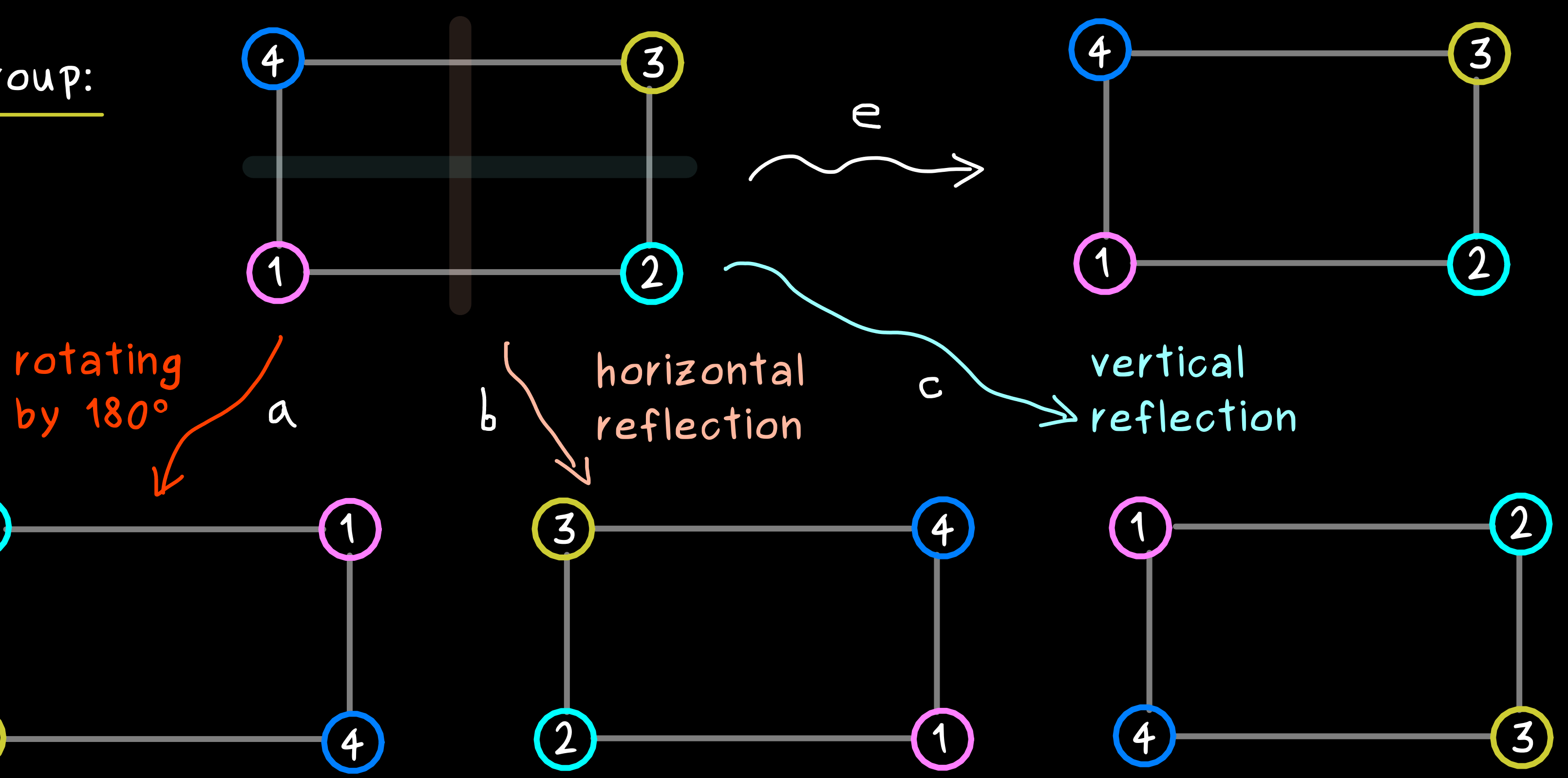
Algebra - Part 11

Recall subgroups: $(G,\circ) \rightsquigarrow H \subseteq G$, $(H,\circ)$ group $\rightsquigarrow$ H subgroup of G $\rightsquigarrow$ $H \leq G$

Proposition: $(G,\circ)$ group, $H \subseteq G$ non-empty subset.

$$H \leq G \iff \begin{cases} a \circ b \in H & \text{for all } a,b \in H \\ a^{-1} \in H & \text{for all } a \in H \end{cases}$$

Klein four-group:



$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\rightsquigarrow$ associativity ✓

$(G,\circ)$ with $G = \{e,a,b,c\}$ and $\circ$ satisfying the table above defines the so-called Klein four group, called K_4.

Proposition: Let $(G,\circ)$ be a group with $\mathrm{ord}(G) < \infty$, $H \subseteq G$ be a non-empty subset.

Then: $H \leq G \iff a \circ b \in H$ for all $a, b \in H$

Proof: $(\Rightarrow)$ ✓ $(\Leftarrow)$ $(H,\circ)$ semigroup of finite order and both cancellation properties hold

$$\begin{pmatrix} a \circ x = a \circ y \Rightarrow x = y \\ x \circ b = y \circ b \Rightarrow x = y \end{pmatrix}$$

$\xRightarrow{\text{part } 6}$ $(H,\circ)$ is a group $\square$

Example: $G = \{e, a, b, c\}$ Klein four-group.

subgroups: $H_1 = \{e\}$, $H_2 = \{e, a\}$, $H_3 = \{e, b\}$, $H_4 = \{e, c\}$, $H_5 = G$

$\leadsto$ we have 5 subgroups