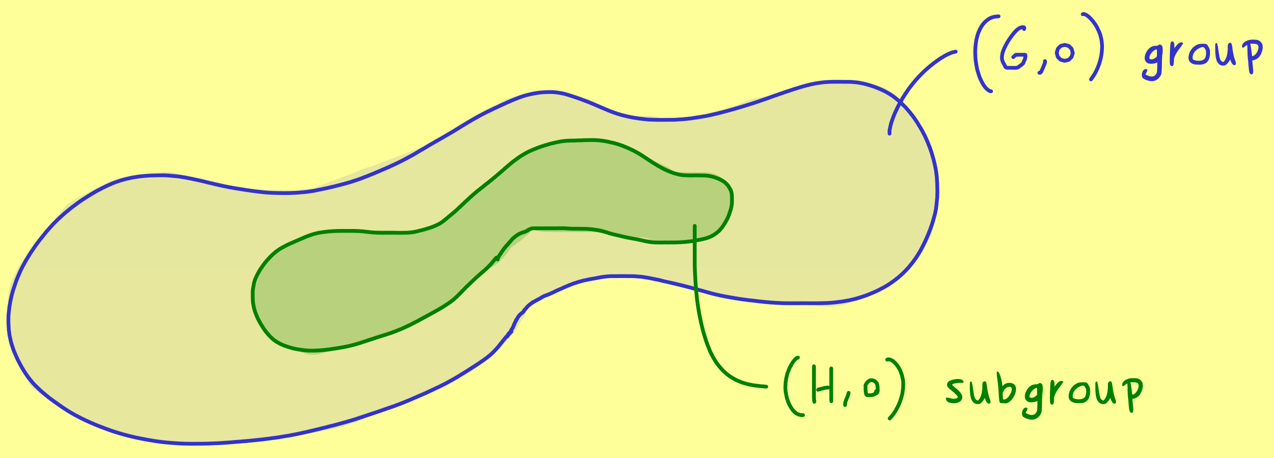
Algebra - Part 10

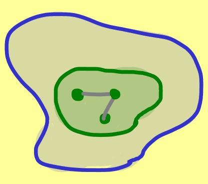


Example: $(\mathbb{R},+) \leftsquigarrow (\mathbb{Z},+)$ subgroup

Definition: Let $(G,\circ)$ be a group. A non-empty subset $H \subseteq G$ is called a subgroup of G if $(H,\circ)$ forms a group.

We get a group homomorphism: $\varphi: H \longrightarrow G$, $x \longmapsto x$, $\quad \varphi(a \circ b) = \varphi(a) \circ \varphi(b)$

$$\Longrightarrow \quad \varphi(e_H) = e_G, \quad \varphi(e_H) = e_H$$



Proposition: Let $(G,\circ)$ be a group and $H \subseteq G$ be a non-empty subset.

Then: H is a subgroup of G $\iff$ $\begin{cases} a \circ b \in H & \text{for all } a,b \in H \quad (*) \\ a^{-1} \in H & \text{for all } a \in H \quad (**) \end{cases}$

Proof: $(\Rightarrow)$ Assume $(H,\circ)$ form a group.

$\Longrightarrow \ \circ: H \times H \to H$ is well-defined! $\Longrightarrow (*)$ ✓

Neutral element in H is the same as the neutral element in G:

$e = x^{-1} \circ x \overset{\text{inverses are unique}}{\Longrightarrow} x^{-1} \in H$ for all $x \in H \Longrightarrow (**)$ ✓

$(\Leftarrow)$ Assume $(*), (**)$. Since $a \circ b \in H$ for all $a, b \in H$,

$\circ: H \times H \to H$ is well-defined!

associative! $(G$ is a group$)$

Choose $a \in H \overset{(**)}{\Longrightarrow} a^{-1} \in H \overset{(*)}{\Longrightarrow} a \circ a^{-1} = e \in H$

$\Longrightarrow (H,\circ)$ is a group $\square$

Example: (a) $(G,\circ)$ group. $\{e\}$ is subgroup of G, G is subgroup of G $\Big\}$ trivial subgroups

(b) $(\mathbb{Z},+)$ group, $m \in \mathbb{N}$. $m\mathbb{Z} := \{ m \cdot k \mid k \in \mathbb{Z} \} \subseteq \mathbb{Z}$

$\Longrightarrow (m\mathbb{Z}, +)$ subgroup of $(\mathbb{Z},+)$

Recall: $\mathbb{Z}/m\mathbb{Z}$ is a group $\rightsquigarrow$ general construction G/H