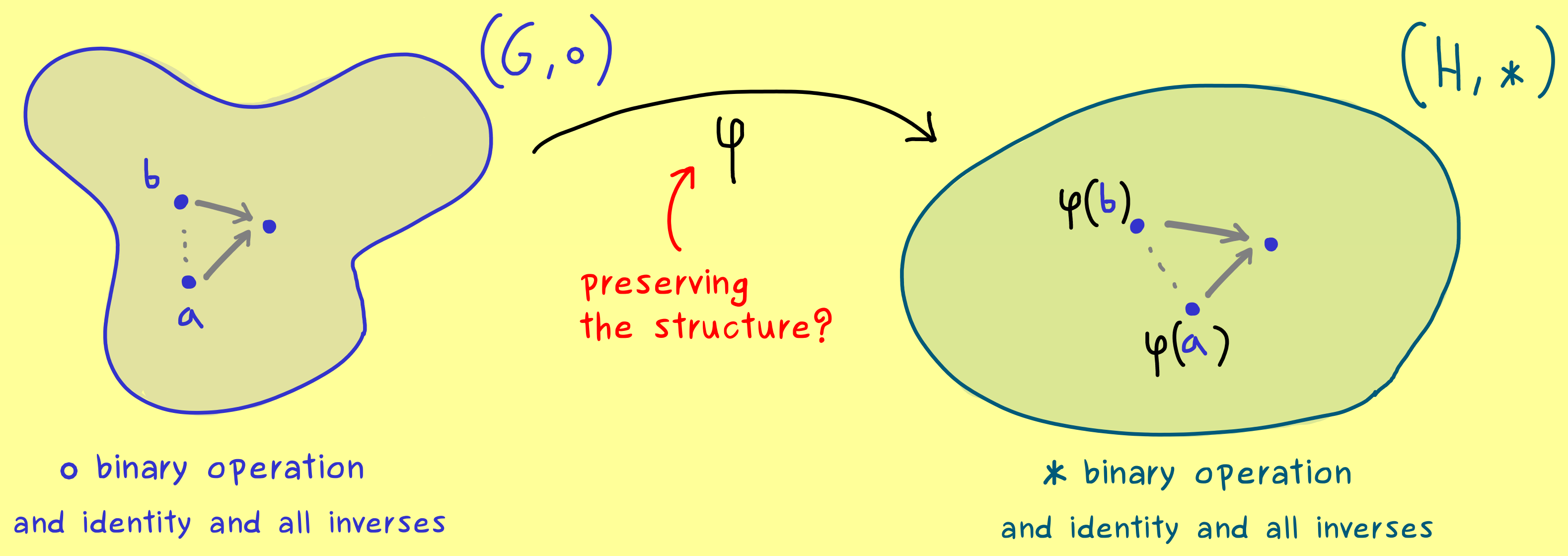
Algebra - Part 9



Definition: $(G, \circ), (H, *)$ groups. A map $\varphi: G \to H$ is called a group homomorphism if $\varphi(a \circ b) = \varphi(a) * \varphi(b)$ for all $a, b \in G$.

Example: $(G, \circ) = (\mathbb{R}, +)$, $(H, *) = (\mathbb{R} \setminus \{0\}, \cdot)$.

$$\varphi: G \to H, \quad x \mapsto e^x \quad \Longrightarrow \quad \varphi(x+y) = e^{x+y}, \quad \varphi(x) \cdot \varphi(y) = e^x \cdot e^y$$

Properties: A group homomorphism satisfies:

(1) $\varphi(e_G) = e_H$ (identity is sent to identity)

(2) $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in G$.

Proof: (1) $\varphi(e_G) = \varphi(e_G \circ e_G) = \varphi(e_G) * \varphi(e_G)$

$$\Longrightarrow e_H = \varphi(e_G)^{-1} * \varphi(e_G) = \varphi(e_G)^{-1} * \big(\varphi(e_G) * \varphi(e_G)\big)$$

$$= \underbrace{\big(\varphi(e_G)^{-1} * \varphi(e_G)\big)}_{= e_H} * \varphi(e_G) = \varphi(e_G)$$

(2) $e_H = \varphi(e_G) = \varphi(a^{-1} \circ a) = \varphi(a^{-1}) * \varphi(a)$

$$\overset{\text{inverse unique}}{\Longrightarrow} \varphi(a)^{-1} = \varphi(a^{-1}) \qquad \square$$