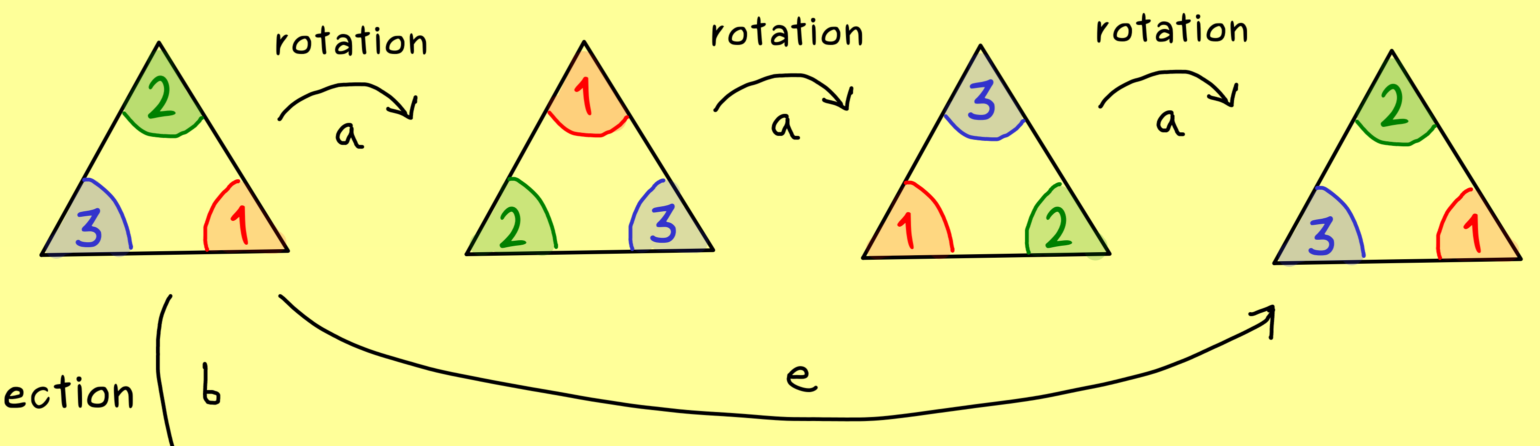
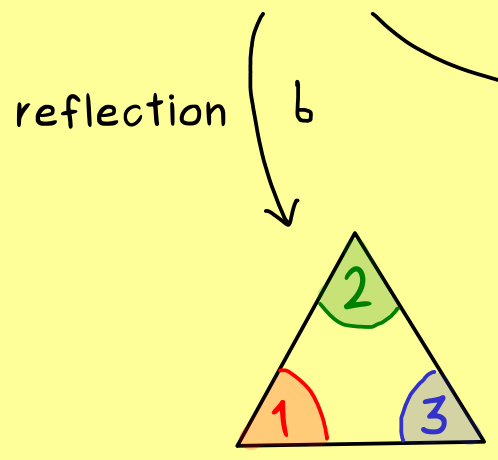
Algebra - Part 7

Group:



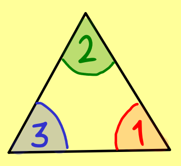


$\rightsquigarrow$ $b^2 = e \Rightarrow b^{-1} = b$

$a^3 = e \Rightarrow a^{-1} = a^2$

$ab \overset{?}{=} ba$

symmetry operations $\longleftrightarrow$ permutations of $\{1, 2, 3\} =: X$



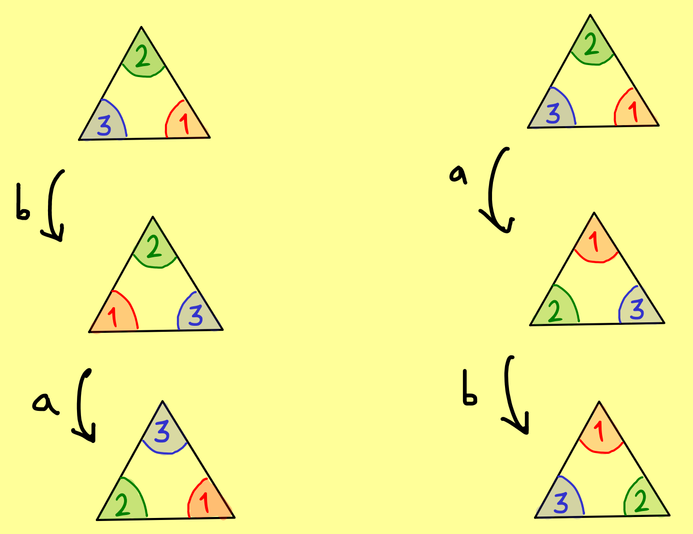
$S_3 := \{ f : X \to X \mid f \text{ bijective} \}$

$\hookrightarrow$ symmetric group

Example:

$f_b(1) = 3$	$f_a(1) = 2$
$f_b(2) = 2$	$f_a(2) = 3$
$f_b(3) = 1$	$f_a(3) = 1$

$\Rightarrow (S_3, \circ)$ composition of maps



We get:

$(f_a \circ f_b)(1) = 1 \quad , \quad (f_b \circ f_a)(1) = 2$

$(f_a \circ f_b)(2) = 3 \quad , \quad (f_b \circ f_a)(2) = 1$

$(f_a \circ f_b)(3) = 2 \quad , \quad (f_b \circ f_a)(3) = 3$

$\Rightarrow$ not commutative!

Definition: A group $(G, \circ)$ (or semigroup) is called abelian or commutative if $a \circ b = b \circ a$ for all $a, b \in G$.

Examples: $(\mathbb{Z}, +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{R}, +)$, $(\mathbb{C} \setminus \{0\}, \cdot)$ are abelian.

General example: $G = \{a, b, e\}$

group with three elements

$\circ$	a	b	e
a	a^2	$\square$	a
b	$\square$	b^2	b
e	a	b	e

1st case: $a^{-1} = b$, $b^{-1} = a$ $\Rightarrow$ $a \circ b = e$

$b \circ a = e$ $\rightsquigarrow$ abelian group

2nd case: $a^{-1} = a$, $b^{-1} = b$ $\Rightarrow$ $(b \circ a) \circ (a \circ b) = b \circ \underbrace{a^2}_{=e} \circ b$

$= e$

$\Rightarrow \underbrace{(a \circ b)^{-1}}_{= a \circ b} = (b \circ a)$ $\rightsquigarrow$ abelian group

Non-abelian group: Symmetric group S_3 : $|S_3| = 3! = 6$ $\Big\}$ order 6

$\shortparallel$

Dihedral group D_3