The Bright Side of Mathematics

Algebra - Part 6

$(S, \circ)$ semigroup. Let's write: $ab := a \circ b$

neutral element + all inverses

group

Fact: Let $(G, \circ)$ be a group and $a, b, x, y \in G$. Then:

$ax = ay \implies x = y$ (left cancellation property)

$xb = yb \implies x = y$ (right cancellation property)

Proof: $x = \underset{\text{neutral element}}{xe} = x(bb^{-1}) = (xb)b^{-1} = (yb)b^{-1} = y(bb^{-1}) = y$

Definition: $(S, \circ)$ semigroup (or group).

The order of S is the number of elements in S:

$$\text{ord}(S) := \begin{cases} |S| = \# S & \text{if } S \text{ is finite} \\ \infty & \text{if } S \text{ is not finite} \end{cases}$$

Lemma: Let $(S, \circ)$ be a semigroup. Then:

$(S, \circ)$ is group $\iff$ $\forall a, b \in S \; \exists x, y \in S : \; ax = b, \; ya = b$

Proof: ($\Rightarrow$) Assume $(S, \circ)$ is a group. For given $a, b \in S$, set:

$x = a^{-1}b$, $y = ba^{-1}$

($\Leftarrow$) For given $a \in S$, there are $x, y \in S$ with $ax = a$, $ya = a$.

Let's call $e := y$: $ea = a$

Let's take $b \in S$. Then there is $\tilde{x} \in S$ with $a\tilde{x} = b$.

We get: $eb = e(a\tilde{x}) = (ea)\tilde{x} = a\tilde{x} = b \implies e$ left neutral

For given $b \in S$ there is $\tilde{y} \in S$ such that: $\tilde{y}b = e \implies b$ left invertible

$\overset{\text{part 4}}{\implies}$ $(S, \circ)$ is a group $\square$

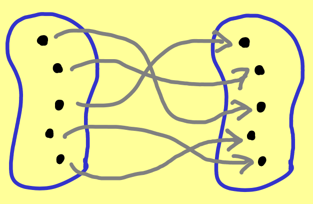
Proposition: Let $(S, \circ)$ be a semigroup with $\text{ord}(S) < \infty$. Then:

$(S, \circ)$ is group $\iff$ both cancellation properties hold

$\left(\begin{array}{l} ax = ay \implies x = y \\ xb = yb \implies x = y \end{array} \right)$

Proof: For any map $f: S \to S$:

f is injective $\iff$ f is surjective



For given $a \in S$, define $f_a : S \to S$ and $g_a : S \to S$ by

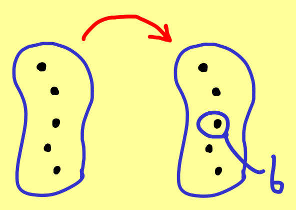
$f_a(x) = ax$, $g_a(x) = xa$.

Then we have: both cancellation properties hold

$\iff \forall a \in S: \; \begin{array}{l} f_a(x) = f_a(y) \implies x = y \\ g_a(x) = g_a(y) \implies x = y \end{array}$

$\iff \forall a \in S:$ f_a and g_a are injective

$\iff \forall a \in S:$ f_a and g_a are surjective



$\iff \forall a \in S:$ for every $b \in S$ there are

$x, y \in S: \; \underset{ax}{\underset{\parallel}{f_a(x)}} = b$ and $\underset{ya}{\underset{\parallel}{g_a(y)}} = b$

$\overset{\text{Lemma}}{\iff}$ $(S, \circ)$ is group $\square$