The Bright Side of Mathematics

Algebra – Part 3

$(S, \circ)$ semigroup $\rightsquigarrow$ $e \in S$ with $e \circ a = a = a \circ e$

Definition: An element $e \in S$ is called

- left neutral (=a left identity) $\quad e \circ a = a \quad$ for all $a \in S$
- right neutral (=a right identity) $\quad a \circ e = a \quad$ for all $a \in S$
- neutral (=an identity) $\quad e \circ a = a = a \circ e \quad$ for all $a \in S$

Example: $S = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$ with $\circ$ given by the matrix multiplication

$\hookrightarrow$ $(S, \circ)$ semigroup

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ left neutral}$$

$$\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ not right neutral}$$

Fact: Let $e \in S$ be left neutral and $\tilde{e} \in S$ be right neutral.

$$\left.\begin{array}{lcl} e \circ a = a & \overset{\text{for } a = \tilde{e}}{\Longrightarrow} & e \circ \tilde{e} = \tilde{e} \\ & & \quad \parallel \\ b \circ \tilde{e} = b & \overset{\text{for } b = e}{\Longrightarrow} & e \circ \tilde{e} = e \end{array}\right\} \Longrightarrow \quad e = \tilde{e}$$

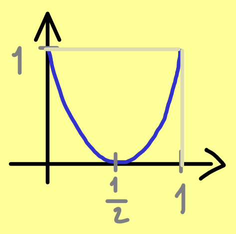
Definition: $(S, \circ)$ semigroup with identity e (the neutral element), $a, b, c \in S$.

- $x \in S$ is called a left inverse of a if $x \circ a = e$ (left invertible)
- $y \in S$ is called a right inverse of b if $b \circ y = e$ (right invertible)
- $z \in S$ is called an inverse of c $\quad z \circ c = e = c \circ z$ (invertible)

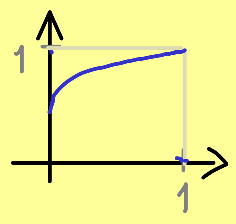
Example: Functions $f: [0,1] \to [0,1]$, $(\mathcal{F}([0,1]), \circ)$ semigroup

Neutral element: $\mathrm{id}: [0,1] \to [0,1]$, $x \mapsto x$

Right invertible: $\tilde{f}: [0,1] \to [0,1]$, $x \mapsto 4(x - \frac{1}{2})^2$



Right inverse of $\tilde{f}$: $g: [0,1] \to [0,1]$, $x \mapsto \frac{1}{2}\sqrt{x} + \frac{1}{2}$



$\hookrightarrow$ $\tilde{f} \circ g = \mathrm{id}$

$g \circ \tilde{f} \neq \mathrm{id}$

Remember:

surjective $\Longleftrightarrow$ right invertible

injective $\Longleftrightarrow$ left invertible