Algebra – Part 2

Definition: Let A be a set.

A map $F: A \times A \longrightarrow A$ is called a binary operation on A.

Instead of $F((a,b))$, we write $a \circ b$ or $a * b$ or $a F b$

or $a \cdot b$ or ab or $a + b$ …

juxtaposition

Closure Law: $a \circ b \in A$ for all $a, b \in A$

Example: $A = \{1, 2, 3\}$, $\circ : A \times A \longrightarrow A$ binary operation defined by:

operation table:

$\circ$	1	2	3
1	3	1	2
2	3	3	1
3	2	2	2

$1 \circ 2 = 1$
$2 \circ 1 = 3$ not equal!

$(1 \circ 2) \circ 3 = 1 \circ 3 = 2$
$1 \circ (2 \circ 3) = 1 \circ 1 = 3$ not equal!

Definition: A pair $(S, \circ)$ where S is a set and $\circ$ is a binary operation on S is called a semigroup if

$$a \circ (b \circ c) = (a \circ b) \circ c \quad \text{for all } a, b, c \in S \qquad \text{(associative)}$$

$$=: a \circ b \circ c$$

Example: Set of functions $\mathcal{F}(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ function} \}$

together with composition $\circ : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R})$:

Take $f_1, f_2, f_3 \in \mathcal{F}(\mathbb{R})$ and define $g = f_1 \circ (f_2 \circ f_3) : \mathbb{R} \to \mathbb{R}$

$h = (f_1 \circ f_2) \circ f_3 : \mathbb{R} \to \mathbb{R}$

$$g(x) = f_1 \circ (f_2 \circ f_3)(x) = f_1((f_2 \circ f_3)(x)) = f_1(f_2(f_3(x)))$$

$$h(x) = ((f_1 \circ f_2) \circ f_3)(x) = (f_1 \circ f_2)(f_3(x)) = f_1(f_2(f_3(x)))$$

$\Longrightarrow$ $(\mathcal{F}(\mathbb{R}), \circ)$ semigroup