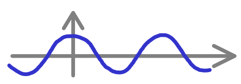
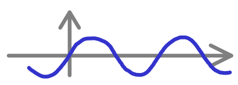
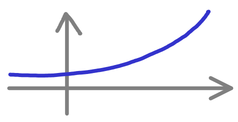
Abstract Linear Algebra – Part 6

subset of $\mathcal{F}(\mathbb{R})$ given by:

$\cos: \mathbb{R} \to \mathbb{R} \rightsquigarrow$ 

$\sin: \mathbb{R} \to \mathbb{R} \rightsquigarrow$ 

$\exp: \mathbb{R} \to \mathbb{R} \rightsquigarrow$ 

$$U := \operatorname{Span}(\cos, \sin, \exp)$$

<u>Question:</u> Is $(\cos, \sin, \exp)$ a basis of U ?

- generating ✓
- linearly independent ?

We have to check: $\underbrace{\alpha_1 \cdot \cos + \alpha_2 \cdot \sin + \alpha_3 \cdot \exp = 0}_{\text{means:}} \Rightarrow \alpha_j = 0$ for all j

zero vector in $\mathcal{F}(\mathbb{R})$

$\hookrightarrow \; 0 : \mathbb{R} \to \mathbb{R}, \; x \mapsto 0$

$$\alpha_1 \cdot \cos(x) + \alpha_2 \cdot \sin(x) + \alpha_3 \cdot \exp(x) = 0(x)$$

for all $x \in \mathbb{R}$

$$\Longrightarrow \begin{cases} \alpha_1 \cdot \cos(0) + \alpha_2 \cdot \sin(0) + \alpha_3 \cdot \exp(0) = 0 \\ \alpha_1 \cdot \cos\left(\frac{\pi}{2}\right) + \alpha_2 \cdot \sin\left(\frac{\pi}{2}\right) + \alpha_3 \cdot \exp\left(\frac{\pi}{2}\right) = 0 \\ \alpha_1 \cdot \cos(-2\pi \cdot 500) + \alpha_2 \cdot \sin(-2\pi \cdot 500) + \alpha_3 \cdot \exp(-2\pi \cdot 500) = 0 \end{cases}$$

$$\Longrightarrow \left\{ \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix}}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right.$$ system of linear equations

Since $\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix} = e^{-1000\pi} + 0 + 0 - 1 - 0 - 0 < 0$,

the system of linear equations is uniquely solvable.

$$\Longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \qquad \Longrightarrow \underbrace{(\cos, \sin, \exp)}_{\mathcal{B}} \text{ basis of } \mathcal{U}$$

Basis isomorphism: $\Phi_{\mathcal{B}} : \mathcal{U} \longrightarrow \mathbb{R}^3$,

defined by $\Phi_{\mathcal{B}}(\cos) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(\sin) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(\exp) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

What about $v : \mathbb{R} \to \mathbb{R}$, $v(x) = 7\cos(x) + 2\exp(x)$

$$\Phi_{\mathcal{B}}(v) = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$$

$$\boxed{\mathcal{U} \text{ is completely represented by } \mathbb{R}^3}$$