



## Abstract Linear Algebra - Part 6

subset of  $\mathcal{F}(\mathbb{R})$  given by:

$$\cos: \mathbb{R} \rightarrow \mathbb{R} \rightsquigarrow \text{graph of } \cos$$

$$\sin: \mathbb{R} \rightarrow \mathbb{R} \rightsquigarrow \text{graph of } \sin$$

$$\exp: \mathbb{R} \rightarrow \mathbb{R} \rightsquigarrow \text{graph of } \exp$$

$$U := \text{Span}(\cos, \sin, \exp)$$

Question: Is  $(\cos, \sin, \exp)$  a basis of  $U$ ?

generating ✓

linearly independent?

We have to check:  $\alpha_1 \cdot \cos + \alpha_2 \cdot \sin + \alpha_3 \cdot \exp = 0 \Rightarrow \alpha_j = 0$  for all  $j$

means:

zero vector in  $\mathcal{F}(\mathbb{R})$

$$\alpha_1 \cdot \cos(x) + \alpha_2 \cdot \sin(x) + \alpha_3 \cdot \exp(x) = 0(x)$$

$$\begin{aligned} \hookrightarrow 0: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto 0 \end{aligned}$$

for all  $x \in \mathbb{R}$

$$\Rightarrow \begin{cases} \alpha_1 \cdot \cos(0) + \alpha_2 \cdot \sin(0) + \alpha_3 \cdot \exp(0) = 0 \\ \alpha_1 \cdot \cos\left(\frac{\pi}{2}\right) + \alpha_2 \cdot \sin\left(\frac{\pi}{2}\right) + \alpha_3 \cdot \exp\left(\frac{\pi}{2}\right) = 0 \\ \alpha_1 \cdot \cos(-2\pi \cdot 500) + \alpha_2 \cdot \sin(-2\pi \cdot 500) + \alpha_3 \cdot \exp(-2\pi \cdot 500) = 0 \end{cases}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right. \text{system of linear equations}$$

$$\text{since } \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix} = e^{-1000\pi} + 0 + 0 - 1 - 0 - 0 < 0,$$

the system of linear equations is uniquely solvable.

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \quad \Rightarrow \mathcal{B} = (\cos, \sin, \exp) \text{ basis of } \mathcal{U}$$

Basis isomorphism:  $\Phi_{\mathcal{B}} : \mathcal{U} \rightarrow \mathbb{R}^3$ ,

$$\text{defined by } \Phi_{\mathcal{B}}(\cos) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \Phi_{\mathcal{B}}(\sin) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \Phi_{\mathcal{B}}(\exp) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

What about  $v: \mathbb{R} \rightarrow \mathbb{R}$ ,  $v(x) = 7 \cos(x) + 2 \exp(x)$

$$\Phi_{\mathcal{B}}(v) = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$$

$\mathcal{U}$  is completely represented by  $\mathbb{R}^3$